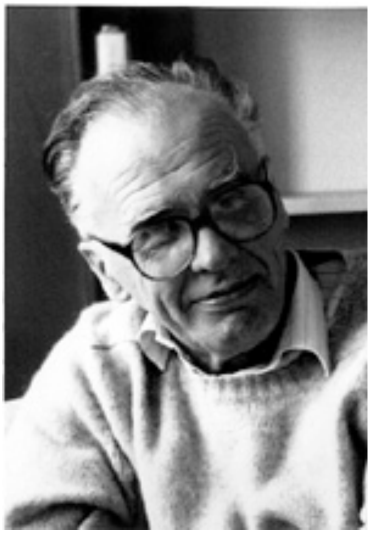


The Abel Prize for 2006 has been awarded Lennart Carleson, The Royal Institute of Technology, Stockholm, Sweden

Professor Lennart Carleson will accept the Abel Prize for 2006 from His Majesty King Harald in a ceremony at the University of Oslo Aula, at 2:00 p.m. on Tuesday, 23 May 2006.

In this background paper, we shall give a description of Carleson and his work. This description includes precise discussions of his most important results and attempts at popular presentations of those same results. In addition, we present the committee's reasons for choosing the winner and Carleson's personal curriculum vitae.

This paper has been written by the Abel Prize's mathematics spokesman, Arne B. Sletsjøe, and is based on the Abel Committee's deliberations and prior discussion, relevant technical literature and discussions with members of the Abel Committee. All of this material is available for use by the media, either directly or in an adapted version.

Oslo, Norway, 23 March 2006

Contents

Introduction

In connection with the awarding of the Abel Prize and on other occasions where mathematical research will be devoted a certain amount of attention from the society-at-large, the question arises as to what this research can be used for. The implied question here is: what use can you and I have of the discoveries that people like Carleson have made. At last year's awarding of the Abel Prize to Peter D. Lax, it was easy to give a popular answer to that question; his work has formed a theoretical basis for the oil production on the Norwegian continental shelf. Stated bluntly, Lax had financed his own Abel Prize and then some, and that was an answer the society-at-large was content with. Here we had someone from this somewhat nerdy world of mathematicians who had done something that was practically useful and hence well worth a prize!

This year's prize winner is not in the same category as Peter D. Lax. Lennart Carleson is a highly theoretical mathematician, though not to be sure of the most abstract sort, but his work concerns issues of primarily theoretical interest. The main result of "convergence almost everywhere for Fourier series of quadratic integrable functions" has little practical utility for some of us in our busy everyday lives. The same can be said about "the existence of a strange attractor for the Hénon map". The words alone are enough to frighten away even the most determined listener.

However, both of these topics also have a more practical side than the one with which Carleson has been occupied. The first topic comes under Fourier analysis, a 200-year-old mathematical tool, which is the be-all and end-all of all engineering. Without Fourier analysis, we would have no cars, TV, tall buildings or vaccines. Indeed, our everyday lives might very well show some resemblance to that of our ancestors in Niels Henrik Abel's time.

The second topic is called dynamic systems. Dynamic systems and modern weather forecasting go hand in glove. Without the theory of dynamic systems, we would be back to traditional weather signs, low-flying swallows and "Red sky at morning; sailors take warning". That's probably good enough for most purposes, but limited meteorological knowledge has resulted in many a grieving fisherman's widow.

To put it somewhat provocatively, we can offer the following historical summary: *Fourier developed Fourier analysis 200 years ago. This mathematical milestone has been crucial to our technological development. Nearly 160 years later, Carleson proved that Fourier was right; the theoretical foundation on which he intuitively based his theory holds water. The engineers who have played with various designs for many generations have finally been given their theoretical seal of approval.*

So, with the awarding of the Abel Prize for 2006 to "the problem solver" of age-old theoretical problems, Lennart Carleson from our sister country, we need once again to man the barricades in defence of theoretical mathematics, for the use of society's revenue on people who sit and ponder tough old problems that do not have an ounce of practical utility.

That defence consists of only one argument, namely: time and again, history has shown that applied and theoretical mathematics are closely related. New advances in the one area are always motivated by experiences from the other. Theoretical mathematical research helps provide the basis for the knowledge that is needed in order to prevent climate disasters, to prevent collapses in the world economy and to develop vaccines against AIDS and avian flu.

This year's Abel Prize winner formulates this as follows in an article: "*... Mathematics in science can be compared to a tree. The roots and trunk are the mathematicians. This is where concepts and methods are generated; we study how various concepts are related to each other. From the trunk, the tree branches out, starting with the main branches, such as fluid theory or probability theory. The fluid will be treated differently, for example, if it concerns a gas, which then little by little results in weather forecasts or if it concerns fluids that are different if they have a low or high viscosity, are hot or cold, electrically charged or magnetised. Everything leads to different theories and specialists, but they are all in the common language of mathematics. Thus, the understanding of this language and the ability to passively or actively handle it is fundamental in a modern society and can be said to be the basis for a country's success. This expertise must embrace all levels.*"

The task of the mathematician is to do mathematics, just as historians study history and politicians practice politics. When the society is simultaneously equipped with clever minds that know how to build bridges, great things can happen.

The Abel Prize is a prize in mathematics. It is awarded to the most worthy mathematician, without considering what benefit the work of the person in question has had for you and me in our everyday lives. The prize is meant to encourage intellectual activity, curiosity and innovation, and it is a recognition that the humanly generated universe of thoughts and ideas that is mathematics is important to human development. Lennart Carleson is a very good example of precisely this idea.

Who is Lennart Carleson?

Lennart Axel Edvard Carleson was born on 18 March 1928 in Stockholm. He studied at Uppsala University, where he took his doctoral degree in 1950, under the guidance of the great Swedish mathematician Arne Beurling. At the mere age of 26, he became a professor at the University of Stockholm, but he returned a year later to Uppsala and a professorship there. Since then he has held professorships at both the University of California, Los Angeles and at the Royal Institute of Technology in Stockholm.

In the period 1968-84, he was director of the Institut Mittag-Leffler at Djursholm, just North of Stockholm. Gösta Mittag-Leffler built this magnificent building at the end of the nineteenth century as a residence, a library and a place where the cultural and academic elite could gather. After Mittag-Leffler's death in 1927, it grew rather quiet under the trees in the park at Djursholm. Carleson, however, recognised the place's potential, arranged the financing and laid the foundation for the Institut Mittag-Leffler that the international community of mathematicians knows today, a centre where researchers from all over the world line up to conduct their research for brief or more extended periods of time.

For 23 years from 1956 to 1979, Carleson was editor of Acta Mathematica, a journal with a long, illustrious history, which is based at the Institut Mittag-Leffler.

In the period 1978-82, Carleson was president of the International Mathematical Union. He worked hard to have the People's Republic of China represented in the Union, a major political affair at the time.

Carleson has been an invited guest speaker three times at the International

Mathematics Conference, and on one of these occasions he was a plenary speaker. This is regarded as a highly prestigious honour in the international community of mathematicians.

Carleson has honorary doctorates at a number of universities and has been elected a corresponding member of a number of Academies of Science and Letters, including the Norwegian Academy of Science and Letters and the Royal Norwegian Society of Sciences and Letters. He has been awarded a number of distinctions for his work, e.g. the Leroy P. Steele Prize from The American Mathematical Society in 1984, the Wolf Prize in 1992, the Lomonosov Gold Medal from the Russian Academy of Sciences in 2002, the Sylvester Medal from the Royal Society of London in 2003 and now the Abel Prize as the current pinnacle in 2006.

Why has he been awarded the Abel Prize for 2006?

The basis in short for the committee's decision is:
for his deep, fundamental contributions to harmonic analysis and to the theory of smooth dynamic systems.

Somewhat (extremely) simplified, we can divide mathematicians into two categories, theory builders and problem solvers. Most mathematicians have a little bit of both in them, but some are more typically one than the other. Carleson fits well into the category of "problem solver". He has made his reputation taking on old, difficult problems, which he has then managed to solve, partly by means of extremely complicated methods. The committee has a rather poetic way of explaining this:

"*Carleson is always far ahead of the crowd. He concentrates on only the most difficult and deep problems. Once these are solved, he lets others invade the new kingdom he has discovered, and he moves on to even wilder and more remote domains of Science.*"

The committee singles out three special problems that Carleson has solved, one of which is ranked above the other two. That is a problem in the field of harmonic analysis, formulated by the Frenchman Jean Baptiste Joseph Fourier in 1807. It concerns the possibility of describing random functions by means of simple wave functions. Fourier was often rather vague and imprecise in his formulations, and it was the Russian mathematician Lusin who precisely defined the problem. He wrote in a work in 1913 that he assumed that the result was true, but that he was unable to prove this assertion. The problem was therefore given the name of Lusin's conjecture. Despite persistent efforts, no one managed to proof this conjecture until Carleson made his breakthrough in 1966 and converted Lusin's conjecture into Carleson's theorem about "the convergence almost everywhere of Fourier series of quadratic integrable functions".

The other two problems that the committee emphasizes in its decision are the "Corona problem" and a problem in dynamic systems related to the Hénon map. The Corona problem is a pure mathematics problem that deals with functions defined on a circular disk. To what extent can we describe what will happen with these functions on the edge of the disk, when we know what happens in the inner area? The name Corona problem refers to the ring of light that is seen around the eclipsed solar disk during a total solar eclipse. Carleson's result has nothing to do with astronomy; mathematicians (in this case the Japanese mathematician Kakutani in the beginning of the 1940s) gave the problem its name by association with a better-known phenomenon.

The Hénon map is named after the astronomer Michel Hénon and refers to work from 1976. In both astronomy and meteorology, it has proven to be expedient to describe phenomena with a system of mathematical models known as dynamic systems. A difficult problem in this theory is to determine whether a system has a so-called “strange attractor”.

The Hénon map describes a way of jumping from point to point in a plane. When we start at a point, several things can happen. We can jump more and more in the direction of a particular point, we can end up on a course where we jump around and around among a finite number of points or we can disappear into infinity. However, it is also possible in the Hénon map that we end up in an area where we do not escape again, but that *within* this area we experience an apparently chaotic behaviour, we jump here and there and back and forth, with the only regularity being that we actually remain within this area. Such an area is called a strange attractor: attractor because the jumps have a tendency to remain within the area, strange because the map displays a strange or chaotic behaviour after we have entered the area.

A computer can easily make the (thousands of) calculations that are necessary in order to visualise the problem, but the computer can never come up with a formal and theoretical proof of the existence of an attractor. In 1991, Carleson and his countryman Benedicks proved that the Hénon map has a strange attractor. This was in fact the first proof that was given for the existence of a strange attractor.

When Carleson began to show interest in dynamic systems in the 1980s, this was a new field of research for him. That a mathematician after the age of 50 was willing to throw himself into something completely new is already rather unusual, but that in the course of a relatively short period of time he was able to solve one of the most challenging problems must be regarded as nothing short of sensational. It does not exactly lend credence to the myth that mathematics is a young man’s game!

The committee's decision is also partly based on Carleson's scientific policy work, though this is not one of the main reasons why he is being awarded the Abel Prize. Throughout his entire career as a mathematician, Carleson has shown great interest in the role of mathematics in the society-at-large. He has thrown himself into the debate on mathematics in school and like many of his colleagues in other countries has expressed concern over the decline in mathematical skills. As president of the International Mathematical Union (IMU) in the period 1978-82, he worked hard to have the People's Republic of China represented in the international society of mathematicians. Back home in Sweden in the 1970s, he built up the Institut Mittag-Leffler into one of the world's most attractive mathematics laboratories, a centre where mathematicians from all over the world can come together in a secluded atmosphere and work on mathematical problems of current interest.

The committee concludes their explanation by noting Carleson's broad range of interests and his important role as both an expert in his field and a spokesman on scientific policy matters: *Lennart Carleson is a brilliant scientist with a broad vision for mathematics and for the role of mathematics in the global community.*

Popular presentations of Carleson's results

Convergence of Fourier series

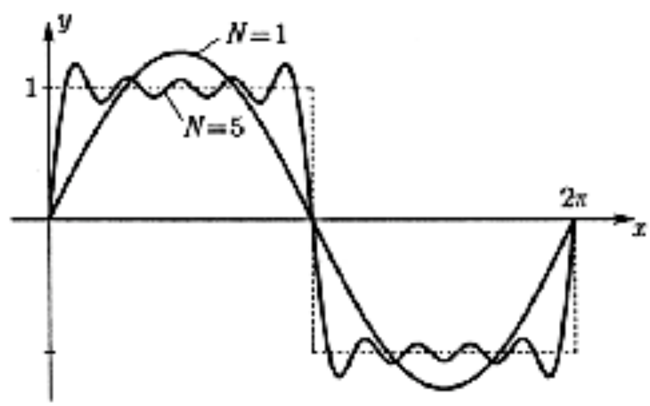
One way of describing Carleson's convergence theorem for the Fourier series of a quadratic integrable function is through sound waves. We can identify a function with a sound, in that the graph of the function describes the vibrations in a membrane. A complex function will normally make a rather noisy sound, whereas a smooth and regular wave function will yield a tone that is clear as a bell, like the sound of a tuning fork.

In this context, the sound from musical instruments may be represented as particular compositions of these smooth, pure tuning-fork waves, known as harmonic vibrations in mathematical terminology. The instruments' overtone patterns, or their individual sound pictures, are characteristic sums of pure harmonic vibrations with wavelengths that are integer products of a fundamental wavelength. Some wind instruments have relatively few prominent overtones, whereas the violin, for example, is specifically distinguished by its great abundance of overtones.

Fourier's problem can be formulated in this setting as follows: *Can an orchestra, if possible with an infinite number of random small instruments, play every conceivable sound?*

With a relatively high degree of approximation, we can say that Carleson in his work from 1966 gives an affirmative answer to this question and that he actually furnishes a rigorous mathematical proof for his assertion.

The figure provides a little insight into how this process of approximation takes place. In this case, we start with a function, illustrated by the dotted line, that has a value of 1 in the interval from 0 to π and of -1 between π and 2π. The first approximation is by a pure harmonic vibration, designated with N=1. We see that this approximation is not very good. If we include 5 terms, drawn in as the curve N=5, we see that the approximation becomes much better. We can continue in this way; the more terms we include, the more the new curve will resemble the original step function curve.



The precise formula for the N=5 curve in this example is

$$y = \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right)$$

and we can easily imagine how we can continue the approximation with more and more terms. An interesting point with this curve in particular is the two “horns” that develop just to the right of x=0 and just to the left of x=π. It turns out that no matter how many terms we use we will not be able to get rid of them. They will gradually become more and more slender, but they will not disappear. This phenomenon is called the Gibbs phenomenon and turns up in a number of similar examples.

We shall not attempt the proof of Carleson's convergence result here; we take the committee's warning seriously: "*The proof of this result is so difficult that for over 30* *years it stood mostly isolated from the rest of harmonic analysis. It is only within the past decade that mathematicians have understood the general theory of operators into which this theorem fits and have started to use Carleson's powerful ideas in their own work."*

The Corona theorem and Carleson measure

In mathematical literature, the word *theorem* is synonymous with the perhaps more easily understandable *main result*. These main results tend to be given names, *Fermat's Last Theorem, Abel's Addition Theorem, the Fundamental Theorem of Algebra* or more everyday names like *the Triangle Inequality, the Spectral Theorem* or the *Corona theorem*.

Carleson's Corona theorem refers to the solar corona, the ring of glowing matter around the sun that can only be observed during total solar eclipses. Carleson has never been an expert on the sun's interior (or exterior) life, but he has proved a difficult theorem that has been given its name from the sun. It was the Japanese mathematician Kakutani who came up with a conjecture that became known as the Corona problem in the beginning of the 1940s. A conjecture is not the same as a theorem; a conjecture has not been proven! This is a trick that mathematicians occasionally resort to. In some situations, they are convinced that a result is correct, even though they cannot manage to proof it. All examples confirm it, and in many special cases the result can actually be formally proven, but they are unable to come up with the final reasoning that is necessary in order to pronounce the result proven. What do they do then? Put everything away in a drawer and try to forget it? No, they pose a conjecture that they publish in the same way as they would fully proven results, and then they call it a conjecture. If he/she is lucky, posterity will name the conjecture after the conjecturer, even if someone completely different actually manages to prove the result.

This then was the fate of the Corona problem, a conjecture everyone believed, but no one could prove. What was it about? The Corona problem considers certain functions defined in a circular disk. The edge of this disk is a circle. If these functions behave properly within the circle, how many curls can they then "come up with" in the actual circle? Carleson's theorem gives an answer to this question. And the analogy to the corona? The circular disk is the sun, and what occurs at the edge, i.e. of the circle, corresponds to the corona.

This result of Carleson is also an example of how the solution of a problem has had an effect on other problems. In Carleson's proof of the Corona problem, he introduces a measure. In this context, a measure is a way of assigning a positive number to a given set. For example, we can define a measure for a series of numbers by assigning an interval its length. Or that a set in a plane is assigned its area. Carleson needed to measure the length of certain curves he constructed on the circular disk and introduced a measure for this purpose. In posterity, this measure has obviously been called the Carleson measure, and it has proven to be an unusually useful aid in many fields of mathematics.

The Hénon map

The most recent work that the committee singles out in its decision to give the Abel Prize to Lennart Carleson originates in the period 1985-1991 and culminates in Carleson and Benedick's dissertation from 1991, where they prove that the Hénon map has a strange attractor. This work lies within the field of dynamic systems. In order to get an insight into what it is about, we shall go back to 1960, to MIT in the USA, where the meteorologist Edward Lorenz was involved in creating good weather models, as meteorologists tend to do. Lorenz had what by today's standards we would call an extremely primitive computer to help him perform the enormous number of calculations that were required in order to predict the weather. Roughly speaking, modern weather forecasting entails considering the physical laws that apply and the initial conditions we have for wind, humidity, pressure etc. right now. Using this description, we calculate the magnitude of these same parameters a short time interval later, then another time interval after that, another after that, etc. until we end up with a forecast of tomorrow's weather. Lorenz had to simplify the whole model down to three parameters, he gave each of these three their value and then he began to “crank the machine”, i.e. make repeated calculations.

The story goes that Lorenz tried one day to continue a run he had started the day before. He began about half way as far as he had come, entered the relevant numbers and started the machine. At first, everything agreed with his observations from the day before, but suddenly the values began to deviate from the previous day's numbers, at first only a little, but then the deviation accelerated rapidly and before he knew it, the model had predicted something completely different from what it had done the day before. How could this happen? The equations were the same, the starting point was the same, the computer was the same, yet the response was different?

The explanation was that they were not the same values. Lorenz rounded off the fourth decimal place when he started on the second day. This meant that the initial conditions were slightly different, but could a difference of one 10,000th of a per cent cause such a catastrophe? We usually take it for granted that a small difference in input gives a small difference in output, but this was not the case here. The reason was that the process was based on repetitions where the previous result became the next premise. A small deviation which is slightly magnified at each step, will eventually lead us into the unknown after many steps. Lorenz had discovered the phenomenon that has come to be known as the butterfly effect in meteorology, namely that a single flap of the wings of a butterfly in Beijing in March can cause the August hurricanes in the Atlantic Ocean to follow a completely different course!

We shall set aside all of the meteorological and other physical consequences of Lorenz’s discovery and focus on its mathematical content. By making use of powerful computers, it was not difficult to create illustrations of the Lorenz system; but this did not give us any particular insight into the mathematical structure, nor did it look as if anyone would be able to provide that insight.

In 1976, the astronomer Michel Hénon presented a simplified version of Lorenz's system. Hénon's discrete dynamic system had two important ingredients. It was much easier to calculate with than the Lorenz system, and like the Lorenz system it had a strange attractor. The Hénon system is described with a map T of a plane into itself expressed by the rule

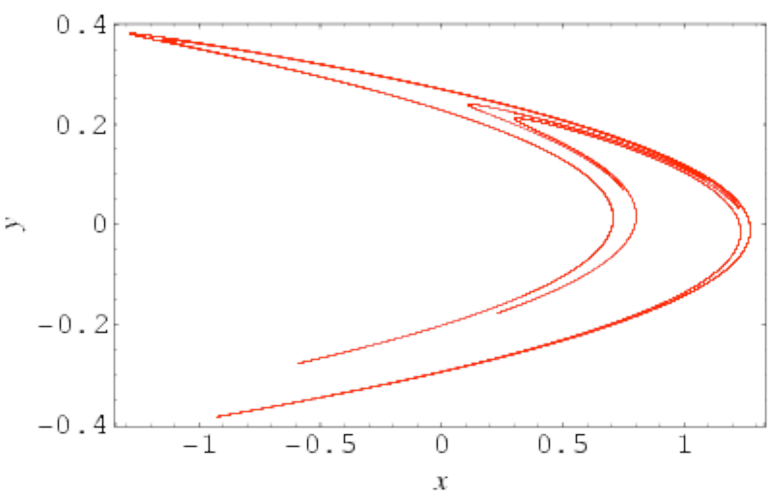
$$T(x, y) = (1 + y - 1.4x^2, 0.3x)$$

(it is okay to use other coefficients instead of 1.4 and 0.3, but these are the ones that are usually used in examples). According to this rule, the point (0, 0) is plotted at the point (1, 0), the point (1, 0) is plotted at the point (-0.4, 0.3), which in turn is plotted at the point (1.076, -0.12), etc. In this case, we end up inside the attractor, the curve that is illustrated in the figure below. The same is true if we begin with the point (0, 0.2918), but not if we begin with (0, 0.2919). In the latter case, we will rapidly disappear far out toward infinity. This is easy to program in a spreadsheet. Enter the following in the four squares at the upper left of the spreadsheet:

◆	A	B
1	0	0
2	=1+B1-1,4*A1*A1	=0,3*A1

Now copy the fields A2 and B2 below in columns A and B as far as you want to go, for 10,000 rows if you like. If we now label all of the fields A1:B10000 and print the graphic symbol at the top of the spreadsheet, choose XY (scatter) and then the alternative without any connecting lines, then we get a map of the Hénon attractor, roughly like the figure below.

The points in this set are distributed in an apparently unsystematic way: one here, one there. It is only when we approach several thousand points that we begin to see the major contours. It turns out that what we think we see is not the whole truth. If we enlarge the slightly thick lines in the attractor, new details steadily emerge: the lines are not single strings, but multiple strings. If we continue to enlarge, we see that the same thing repeats itself, the single strings always split up into even smaller single strings. We see that the attractor has fractal properties.

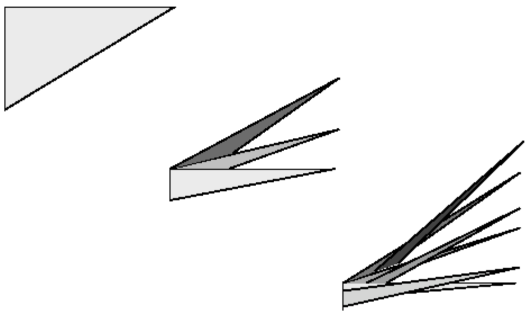


This is where the really difficult question arises. If we make 10,000 calculations in this way, or if we make 10,000,000 calculations, we get about the same picture – more calculations will only reveal more of the fractal structure. But how do we know that we will not suddenly fly off into infinity, as we did after we had made about 35 iterations when the starting point was (0, 0.291807922563607)? Is this really an *attractor*?

Carleson and Benedicks presented a formal proof that the world is exactly as we think it is, that a strange attractor does exist. Nothing unforeseen is going to happen even if we make quadrillions of iterations. Once we are inside this strange set, we will remain there. The problem may seem rather contrived and narrow to a non-mathematician, but this is mathematics, not meteorology or physics, and therefore it is imperative that we know and not just believe.

Kakeya's needle problem

The fourth problem that the committee mentions in its decision is not a problem for which Carleson has contributed an epoch-making breakthrough as was the case in the three above-mentioned problems. However, in collaboration with his student Sjölin, Carleson has proved a result that has turned out to be important in the study of this problem and especially its generalisations. Since the nature of the problem is fairly easy to explain, we shall include a description of it here.



We dip a needle in ink and lay it on a sheet of paper. The problem involves rotating the needle 180 degrees without lifting it from the paper. We are allowed to push it back and forth while we rotate it, roughly as we do when we turn a car around in a parking space. The question is how large the area on the sheet that is coloured with ink will be when we are finished, and particularly if there is a lower limit to this area when we choose an optimal way to carry out the rotation. This problem is known as Kakeya's needle problem, formulated by the Japanese mathematician Kakeya in 1917.

Let us assume that the needle has a length of 1. If we rotate the needle around its midpoint, we will have coloured a space with an area equal to $\pi \cdot 0.5^2 \approx 0.78$. This is clearly not optimal; we can definitely manage to turn inside an equilateral triangle with a height of 1 and with an area approximately equal to 0.58. Better yet would be a hypocycloid (the curve that is traced by a point on a small wheel that rolls around inside a larger wheel) with a diameter of 1. This has an area of 0.39. However, all of these answers are wrong. The correct answer is that the surface can be made as small as we like! This was proved by the Russian mathematician Besicovitch in 1928. His set consisted of a large number of very long and narrow triangles, almost like the branches on a Christmas tree in a child's drawing. It involves driving back and forth a great many times and only turning a little each time.

Current interest is focused on the generalisations of this problem, and this is where Carleson-Sjölin's result for Fourier multipliers enters in as a standard tool.

Precise (mathematical) formulations of Carleson's results

Convergence of Fourier series

Let f be an integrable function defined in the interval $[-\pi,\pi]$. We define the mth Fourier coefficient of f as

$$\hat{f}(m) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-imt}dt$$

The nth partial sum of the function f is given by

$$s_n(x) = \sum_{m=-n}^{n} \hat{f}(n)e^{imt}$$

Theorem (Carleson).

If f(x) is quadratically integrable, then $s_n(x)$ converges toward f(x) almost everywhere.

L. Carleson; On Convergence and Growth of Partial Sums of Fourier Series. Acta Mathematica, Volume 116, pp. 135-157.

The Corona problem

Theorem (Carleson).

*Let **B** be the Banach algebra of bounded analytic functions in the open unit disk in the complex plane under the natural norm. Let $f_1,\ldots,f_n$ be given functions in **B** so that*

$$|f_1(z)| + |f_2(z)| + \cdots + |f_n(z)| \geq \delta > 0$$

*for a real number δ. Then $I(f_1,\ldots,f_n) = $ **B**.*

L. Carleson; Interpolations by Bounded Analytic Functions and the Corona Problem. Annals of Mathematics, Volume 76, No. 3, November 1962, pp. 547-559.

An alternative formulation:

Let ***M*** be the set of maximal ideals in ***B*** (defined above). Since the quotient ***B/m*** for a maximal ideal ***m*** is isomorphic to the complex numbers ***C,*** we can identify ***m*** in a natural way with a homomorphism φ_m: ***B→C*** so that ***m***=ker(φ_m). This induces a natural map π: ***M→D*** of ***M*** into the closed unit disk in the complex plane, given by π(φ)=φ(z), where z is the identity map of the disk. The continuity of φ ensures that $|\varphi(z)| \leq |z| \leq 1$. For each element ω in the interior D^0 of the unit disk, we can form the maximal ideal m_ω consisting of all functions vanishing in ω. Since z-ω lies in m_ω, then

$$\pi(\varphi_{m_\omega})=\varphi_{m_\omega}(z)=\varphi_{m_\omega}(z-\omega+\omega)=\varphi_{m_\omega}(z-\omega)+\varphi_{m_\omega}(\omega)=\omega$$

and we get a natural embedding of the open unit disk D^0 into ***M***. The complement of D^0 is mapped by π into the unit circle $|\omega|=1$, and for each complex number ω of absolute value 1, we can form the fibre $M_\omega=\pi^{-1}(\omega)$. This fibre contains homomorphisms that correspond to "evaluation in ω", where the quotes remind us that the homomorphisms are actually not defined at the boundary. Now, however, we have the following result (taken from K. Hoffman: Banach spaces of Analytic functions, Prentice-Hall, 1962):

*Let f be a function in **B** and let ω be a point on the unit circle. Let $\{\lambda_n\}$ be a sequence of points in the open unit disk D^0 so that $\lambda_n \to \omega$. Assume further that the limit ζ of the sequence $f(\lambda_n)$ also exists. Then there is a complex homomorphism φ in the fibre M_ω so that φ(f)=ζ.*

Thus, a priori there may be a set of "evaluation homomorphisms" on the unit circle, corresponding to various ways of approaching the point ω. It is this manifold that has given rise to the name "the Corona problem". Carleson's result states that D^0 is dense in ***M***, i.e. that the whole corona is contained in the closure of the open disk.

In Carleson's proof of the Corona theorem, he introduces a particular measure, which has been given the name Carleson measure by posterity.

Definition

Let μ be a non-negative measure of the open unit disk $\boldsymbol{D}^0$ in the complex plane and assume that

$$\mu(S) \le C \cdot h$$

for all sets S of the form

$$S = \left\{ re^{i\theta} \middle| r \ge 1 - h, \theta_0 \le \theta \le \theta_0 + h \right\}$$

Then μ is called a Carleson measure.

Carleson needs this definition in order to prove the following result. We let H^p, $1<p<\infty$ be the Banach space of bounded analytic functions f defined in the interior of the complex unit disk $\boldsymbol{D}^0$ under the norm

$$\|f\|_p = \lim_{r\to 1}\left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{1/p}$$

Theorem (Carleson)

Let μ be a non-negative measure of $\boldsymbol{D}^0$. Then μ is a Carleson measure if and only if

$$\int |f(z)|^p d\mu \le C_p \|f\|_p^p$$

for every $f \in H^p$, $1<p<\infty$ and where C_p is a constant.

Existence of a strange attractor for the Hénon map

The Hénon map is defined as the discrete-time dynamic system given by

$$x_{t+1} = 1 - ax_t^2 + y_t$$

$$y_{t+1} = bx_t$$

We call the map *T*. In Hénon's original example, *a=1.4* and *b=0.3*.

Theorem (Benedicks, Carleson).

Let W^u be the unstable manifold of T at its fixed points in x,y > 0. Then for all c < log2, there exists a $b_0 > 0$ such that for all $0 < b < b_0$ there exists a set E(b) of positive one-dimensional Lebesgue measures such that for every a in the set E(b):

(i) There exists an open set U=U(a,b) such that for every z in U,

$$dist(T^\nu(z), \overline{W^u}) \to 0 \text{ when } \nu \to \infty.$$

(ii) There is a point $z_0=z_0(a,b) \in W^u$ such that

a. $\left\{ T^\nu(z_0)^{-} \right\}_{\nu=0}^{\infty}$ *is dense in* W^u*;*

b. $\left\| DT^\nu(z_0)(0,1) \right\| \ge e^{c\nu}$.

M. Benedicks, L. Carleson; The dynamics of the Hénon Map. Annals of Mathematics, Volume 133, No. 1, 1991, pp. 73-169.

In this case, *DT* stands for the Jacobi matrix of *T*.

The Hénon attractor is fractal, smooth in one direction and a Cantor set in another. Numerical estimates yield a correlation dimension of 1.42 ± 0.02 (Grassberger, 1983) and a Hausdorff dimension of 1.26 ± 0.003 (Russel, 1980) for the attractor of the canonical map.

Simply stated, Benedicks-Carleson's result says that the Hénon map has so-called strange attractors for a non-empty (up to and including of positive measure) set of parameter values.

Background material

Jean Baptiste Joseph Fourier (1768-1830)

Joseph Fouriers was the twelfth of a total of fifteen siblings. He showed considerable ability at an early age, in both literature and mathematics, but nevertheless chose to educate himself to become a priest. He was not at all certain that this was the right choice; he may well have felt that mathematics was dearest to his heart. Gradually he also began to develop an interest in politics, and he joined the local revolutionary committee. This was in the years after the French Revolution in 1789, and as is often the case in history, “the Revolution eats its children”. Fourier was arrested and feared for a while that he would end up on the guillotine, but fortunately he was released after a while. In 1795, he was accepted in the first class of students at the École Normale in Paris and was instructed in mathematics by Lagrange, Laplace and Monge, three of the greatest mathematicians in Europe at that time. At the same time, he began to teach at the École Polytechnique without initially doing any particular research of his own.

In 1798, Fourier joined Napoleon's campaign to Egypt as a scientific advisor. The campaign was very successful at first, but in the Battle for the Nile Napoleon’s fortunes took a turn for the worse. Lord Nelson was too strong, and after the defeat Napoleon and his men were nearly held prisoner in Egypt. Fourier made good use of the time and was intensely involved in the work of developing educational and research institutions in Cairo, based on the French model. His main contribution was to organise the work on a description of Egypt's geography and history, and he himself wrote an introduction, which is still regarded to this day as a breakthrough in Egyptology.

Back in Paris, Fourier was nearly forced into politics after a while by Napoleon, who had ascended the throne by then. Fourier would have preferred to practice mathematics, but you did not refuse the Emperor. As Prefect of the Isère-region, however, he had time to do research, and it was in this period (1807) that he published his work on heat conduction in solid materials, where harmonic functions make their appearance. This work was not well received by his contemporaries, but posterity has learned to greatly appreciate Fourier's ideas. In fact, his dissertation from 1807 is the origin of the concept of Fourier analysis, a field that has proven to have enormous significance and that is currently a key component of every engineering student's everyday life. Also found in this dissertation is the problem that Lennart Carleson solved in 1966 and for which he has won the Abel Prize for 2006.

Fourier's life after this dissertation was affected by his somewhat ambivalent relationship to Napoleon and the fact that his 1807 dissertation was in many ways highly controversial. He was given positions of power and influence, but lived in constant danger of falling into disfavour. He was famous, but at the same time controversial among the mathematicians of his day.

Institut Mittag-Leffler

Institut Mittag-Leffler was founded in 1916 by Gösta Mittag-Leffler and his wife Signe, whose maiden name was Lindfors. The operation of the institute was taken over by The Royal Swedish Academy of Sciences in 1919.

Gösta Mittag-Leffler (1846-1927) was an influential Swedish mathematician and business man. Like Lennart Carleson he was educated in Uppsala, where he received his doctoral degree in 1872. After living abroad for many years in Paris, Berlin and Helsinki, he returned to his birthplace of Stockholm and held the first mathematics professorship at the city's new university. As a university professor, Mittag-Leffler had many irons in the fire, e.g. he founded *Acta Mathematica*, a prestigious journal that is still issued under the direction of the Institut Mittag-Leffler.

Throughout his life, Gösta Mittag-Leffler was a bibliophile, and his collection of books eventually became quite impressive. With his wife's inheritance, and later with his own earnings, he could buy almost anything he wanted; and what he did not spend on books, he spent on building himself a little palace at Djursholm, an upper-class suburb just north of Stockholm. In his last will and testament, which he made public on his 70th birthday in 1916, Mittag-Leffler provided the basis for a foundation with the purpose of promoting research in pure mathematics in the Scandinavian countries. This foundation was supposed to run the big library at Djursholm and simultaneously support a research institute with ample access to scholarships for young mathematicians, based on the model of the Pasteur Institute in Paris. In 1916, Mittag-Leffler's wealth was so great that these were highly realistic plans, but the stock market crash in 1922 changed the situation dramatically and by the time of Mittag-Leffler's death in 1927 there was very little left with which to realise his ambitious plans.

After that, Institut Mittag-Leffler remained involuntarily dormant for decades until Lennart Carleson took over as director in 1969. Carleson acquired a new financial base for the institute through contributions from the Wallenberg Foundation and several insurance companies. In addition, contracts were signed for support from the research councils in Sweden, Finland, Denmark and Norway. This made it possible for Carleson to start building up the institute the way Mittag-Leffler had indicated in his last will and testament. One thing that Mittag-Leffler and Carleson had in common was their support of young mathematicians, and in the institute that emerged under Carleson's direction, this group was given special attention. Carleson remained director of the institute until 1984 and handed over an institute that promised to become great.

Today, the institute is one of the most attractive research institutions of its kind in the world, visited by hundreds of mathematicians each year, including many of the most prominent ones. The fields of research at the institute change each year and a steady stream of new mathematicians are given an opportunity to visit Gösta Mittag-Leffler's magnificent library and take part in seminars in the old dining room, where both Ibsen and Strindberg enjoyed the mathematics professor's generosity in bygone days. In the academic year 2001-2002, Lennart Carleson himself was back at the institute, as one of three leaders for a year that focused on *Probability and Conformal Mappings*.

Fourier analysis

Let *f* be a function defined on the real numbers. We say that *f* is periodic with a period T if *f* fulfils the equation *f*(x+T)=*f*(x) for all real numbers x, where T is the smallest number with this property. Typical examples of periodic functions are sin(kx) and cos(kx).

Theorem

Let f be continuous on I=[-π, π]. Assume that the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

converges uniformly to the function f on the interval I. Th*en we have*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \quad n=0,1,2,...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \quad n=1,2,...$$

Definition

The coefficients a_n *and* b_n *are called Fourier coefficients of f, and the series*

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

is *called the Fourier series of f.*

Example

Let *f* be the function that equals 1 in the interval [0, π] and -1 in [-π, 0]. Then we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}\cos(nt)dt = \frac{1}{n\pi}0 + \frac{1}{n\pi}0 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)dt = \frac{1}{n\pi}(1-(-1)) + \frac{1}{n\pi}(-(-1)+1) = \frac{4}{n\pi}$$

when n is odd and b_n=0 when n is even. This gives the Fourier series

$$f \sim \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Discrete dynamic systems

Dynamic systems are often described by differential equations. These equations model the relationships between a physical system's changes and its state. In order to understand a physical system's development in time, it is necessary to be able to integrate the system, i.e. to find solutions for the differential equations, either analytically as compact formulas or numerical by extensive use of computers.

One example of a dynamic system is this model taken from ecology. We consider two species that compete for space in a geographically limited area. The one species lives by eating the other. We let $x=x(t)$ be the population of the predator, while $y=y(t)$ is the population of the prey, both at time t. We can now set up a system of differential equations that describes the trend of the two populations:

$$\frac{dx}{dt} = k_1xy - k_2x$$

$$\frac{dy}{dt} = -k_3xy + k_4y$$

This gives us a dynamic system. The solutions to the system will describe the trend of the two animal populations with time.

In a discrete dynamic system, the time parameter is converted to a discrete quantity. The equations that are included in the system describe the next state as a definite function of the previous state. A simple example is the discrete logistical growth model, given by:

$$x_{n+1} = kx_n(1 - x_n)$$

Let us assume that x_0=0.5 and see what kind of development we get. In the table, we let the constant k assume four different values. We get four different trends; for $k=1.5$ we get a so-called fixed point, i.e. x_n gradually stabilises at a fixed value, in this case 0.3333... . For $k=3.2$ we get a different state of equilibrium, two different values for x_n and regular alternation between them. For $k=3.5$ we get something similar, but this time we alternate among four values. In all of these examples, we observe an attractor, or a set that the system approaches with time. For $k=3.9$, however, we get a completely new situation, the trend becomes chaotic, apparently without any easily recognisable system.

n	*k=1.5*	*k=3.2*	*k=3.5*	*k=3.9*
1	0.3750	0.8000	0.8750	0.9750
2	0.3515	0.5120	0.3828	0.0950
3	0.3419	0.7995	0.8269	0.3355
4	0.3375	0.5128	0.5008	0.8694
5	0.3354	0.7995	0.8749	0.4426
6	0.3343	0.5130	0.3828	0.9621
7	0.3338	0.7995	0.8269	0.1419
8	0.3335	0.5130	0.5008	0.4750
9	0.3334	0.7995	0.8749	0.9725
10	0.3333	0.5130	0.3828	0.1040
11	0.3333	0.7995	0.8269	0.3634
12	0.3333	0.5130	0.5008	0.9022

The logistical growth model is an example of a one-dimensional dynamic system. The Hénon map is an example of a two-dimensional system.

$$x_{n+1} = 1 - ax_n^2 + y_n$$

$$y_{n+1} = bx_n$$

Like the one-dimensional system, this system also has fixed points, i.e. values where $x_{n+1}=x_n$ and $y_{n+1}=y_n$. We find these at:

$$x_n = \frac{1}{2a}(b - 1 \pm \sqrt{(1-b)^2 + 4a})$$
$$y_n = bx_n$$

In Hénon's standard example, the values of the constants are *a=1.4* and b=0.3. For the one fixed point, this gives $x_n \approx 0.63135$ and $y_n \approx 0.18941$. The Hénon map has a strange attractor (proved by Carleson and Benedicks), a set that is such that once the system has entered the attractor, it will remain there, but inside the attractor we have a chaotic trend.

References to the illustrations

Fourier analysis

http://www.falstad.com/fourier/
http://www.jhu.edu/~signals/fourier2/
http://www.univie.ac.at/future.media/moe/galerie/fourier/fourier.html

The Hénon map

http://www.cmp.caltech.edu/~mcc/Chaos_Course/Lesson5/Demo1.html
http://library.thinkquest.org/26242/full/fm/fm12.html
http://www.cs.laurentian.ca/badams/Attractors2D/HenonApplet.html