Calculus of Variations

The calculus of variations goes back to the 17th century and Isaac Newton. Newton developed the theory to solve *the minimal resistance problem* and later *the brachistochrome problem.*

The minimal resistance problem asks for the surface, formed as a curved cylinder, which experiences the minimum resistance when it moves through a homogeneous fluid. The surface moves with constant velocity and in the direction of the cylinder axis.

The brachistochrome problem, originally posed by Johann Bernoulli, is to find the curve connecting two points, on which a bead slides frictionlessly in the shortest time. The bead starts from rest and moves under the influence of a uniform gravitational field only.

The calculus of variations was further elaborated by Leonhard Euler and later by Joseph-Louis Lagrange. The general formulation of the theory is as follows:

Let J be a functional that assigns a value to any function in some function space. Find the function that minimizes J.

One simple example of calculation of variations is the problem of finding the shortest path between two points (x_1, y_1) and (x_2, y_2) in the plane. Let $y = f(x)$ be a function connecting the two points, i.e. $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then the arc length of the curve defined by the function f is given by

$$J(f) = \int_{x_1}^{x_2} \sqrt{1 + f'(x)^2}\, dx$$

We want to minimize the length of the curve, i.e. to find a function f such that J obtains its smallest value.

An outcome of the work on the calculation of variations by Euler and Lagrange is the so-called Euler-Lagrange equation, which gives a general solution to the variation problem. The result says that in order to find the critical function for the functional

$$J(f) = \int_S L(f, f', x)\, dx$$

where L is a function of x, $f(x)$ and the derivative $f'(x)$, and S is the region of integration, one has to solve the Euler-Lagrange equation;

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$$

In the above example the functional L is given by $L(f, f', x) = \sqrt{1 + (f')^2}$. Then we have

$$\frac{\partial L}{\partial f} = 0 \qquad \frac{\partial L}{\partial f'} = \frac{f'}{\sqrt{1 + (f')^2}}$$

and the Euler-Lagrange equation takes the form

$$-\frac{d}{dx}\frac{f'}{\sqrt{1 + (f')^2}} = 0$$

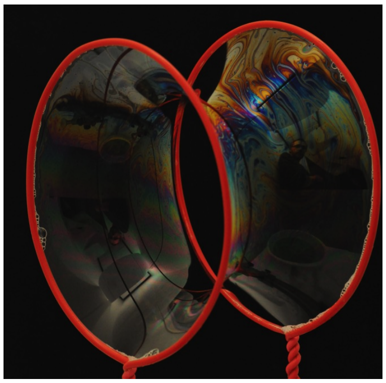
or consequently

$$\frac{f'}{\sqrt{1 + (f')^2}} = c$$

Solving for f' we find that $f'(x)$ is constant, i.e. $f(x) = ax + b$. The solution expresses the fact that the shortest path between to points is the straight line.

Minimal surfaces are illustrations of the calculus of variations in higher dimensions. Let $f : S \to M$ be a (sufficiently smooth) map from a surface S into some manifold M (we may think of M as Euclidian 3-space). Now suppose there are two holes in M and that the surface $f(S)$, being the image of a sphere, enclose the two holes. The problem is to find a surface which fullfill these conditions and has minimal area. Since we are obliged to enclose the two holes, it is not possible to shrink the surface to zero area, so we may hope that there in fact exists a surface of minimal area.

Let us look at an example close to the situation just described. We consider two parallel circles, facing each other, of radius r and distance d. Consider a surface given as the image $u(S)$ of a cylinder S where the boundary circles, ∂S, are mapped to the two parallel circles.



[1] *A soap film connecting two parallel circles.*

If we assume that the surface is symmetric about the axis connecting the two circles, i.e. defined by the revolution of a function $y = f(x)$ along the axis, the area of the "soap cyclinder" is given by

$$A(f) = \int_S 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$$

Applying the Euler-Lagrange equation to the functional $L(f, f', x) = f(x)\sqrt{1 + f'(x)^2}$ and using the Beltrami identity (which gives a simplified version of the Euler-Lagrange equation in the case $\frac{\partial L}{\partial x} = 0$), the minimizing function has to satisfy

$$\frac{d}{dx}\left(f(x)\sqrt{1 + f'(x)^2} - \frac{f(x)f'(x)^2}{\sqrt{1 + (f'(x))^2}} \right) = 0$$

or equivalently

$$\frac{f(x)}{\sqrt{1 + (f'(x))^2}} = c$$

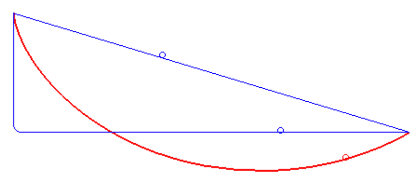
The solution of this differential equation is the catenary curve;

$$y = c\cosh\left(\frac{x}{c}\right)$$

The catenary curve is also known as the hanging cable curve, i.e. the curve that an idealized hanging cable assumes under its own weight when supported only at its ends. Revolution of the catenary about the axis gives a so-called catenoid. Thus the minimal surface attached to parallel circles will take the shape of a catenoid.

[2] *A spider web consists of a lot of hanging cables.*

Let us go back to the brachistochrome problem. Even if the straight line is the shortest path between two points, it is not the fastest. In the beginning we need to gain speed, and even though the consequence is a longer path, the increased speed more than compensate for the increased distance.



The three circles were released from the top point at the same time. The red one will reach the end point first.

The time to travel from a point P to another point Q along a curve $\gamma : y = f(x)$ is given by the integral

$$T(\gamma) = \int_P^Q \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2f(x)}} dx$$

(we assume gravitation acceleration $g = 1$ for simplicity).

The Euler-Lagrange equation and the Beltrami identity give the differential equation of the solution $y = f(x)$;

$$\frac{d}{dx}\left(\frac{\sqrt{1 + f'(x)^2}}{\sqrt{2f(x)}} - \frac{f'(x)^2}{\sqrt{1 + f'(x)^2}\sqrt{2f(x)}} \right) = 0$$

or simplified

$$\frac{1}{\sqrt{1 + f'(x)^2}\sqrt{2f(x)}} = c$$

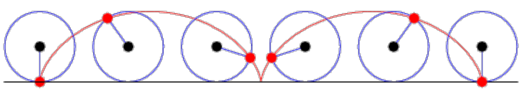
which we can write

$$\left(1 + f'(x)^2\right) f(x) = k^2$$

where $k = \frac{\sqrt{2}}{2c}$. The solution of this equation can be given as the parametrized curve

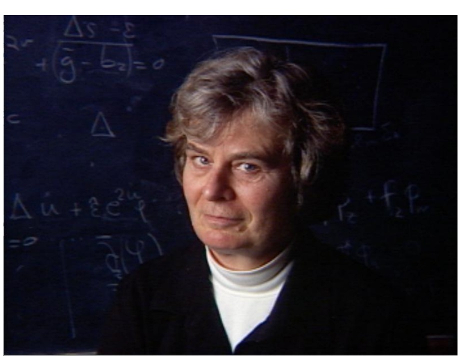
$$x = \frac{1}{2}k^2(\theta - \sin\theta), \quad y = \frac{1}{2}k^2(1 - \cos\theta)$$

This curve is the cycloid, often described as the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping.



The history of the calculus of variations is tightly interwoven with the history of mathematics. Many outstanding mathematicians have contributed to the theory. We have already mentioned Isaac Newton, Johann Bernoulli, Leonhard Euler and Joseph-Louis Lagrange. Later mathematicians as Hamilton, Jacobi, Dirichlet and Hilbert have offered their contributions. In modern times, the calculus of variations has continued to occupy center stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics.

Minimization problems that can be analyzed by the calculus of variations serve to characterize the equilibrium configurations of almost all continuous physical systems, ranging through elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics, string theory, and many, many others.



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