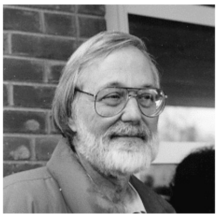
THE ABEL PRIZE



Abel Prize Laureate 2011
John Willard Milnor

Dictionary for a better understanding of the citation of the Abel Commitee

In the citation, the Abel Commitee gives the background for this year´s award of the Abel Prize. The citation is not easily acessible for the average reader. In this note we try to give a (hopefully) more understandable explanation of the terms and concepts in the citation.

Some relevant fields of mathematics
Topology (from the Greek topos, "place", and logos, "study") is a major area of mathematics concerned with shapes, such as curves, surfaces, solids, etc. In topology one studies properties of the shapes that are preserved through continuous deformations, twistings, and stretching of objects. Tearing or gluing, however, is not allowed. One does not care about distances, angles or curvature. A donut, and a mug, with one handle, are topologically equivalent, i.e. from a topological point of view we do not distinguish between the two shapes. Topology is sometimes called rubber band geometry.

Geometry (Ancient Greek, geo- "earth", -metri "measurement") "Earth-measuring" is a branch of mathematics concerned with concepts such as shape, size, relative position of figures, and the properties of space. Geometry is one of the oldest mathematical sciences. The difference between topology and geometry lies in the concept of measurement. The shortest distance between two points on a surface is a geometrical property, but not a topological property. Thus if we stretch the surface such that the distance between two given points are changed, the geometry of the surface is changed, but not necessarily the topology.

Differential topology and ***differential geometry*** are first characterized by their similarity. They both study primarily the properties of differentiable manifolds, sometimes with a variety of structures imposed on them.
From the point of view of differential topology, the donut and the mug are the same. A differential topologist imagines that the donut is made out of a rubber sheet, and that the rubber sheet can be smoothly reshaped from its original configuration as a donut into the shape of a mug without tearing the sheet or gluing bits of it together. This is an inherently global view, though, because there is no way for the differential topologist to tell whether the two objects are the same (in this sense) by looking at just a tiny (local) piece of either of them. He must have access to each entire (global) object.
From the point of view of differential geometry, the mug and the donut are different because it is impossible to rotate the mug in such a way that its configuration matches that of the donut. This is also a global way of attacking the problem. But an important distinction is that the geometer does not need the entire object to decide wether two distinct objects are different. By looking, for instance, at just a tiny piece of the handle, he can decide that the mug is different from the donut because the handle is thinner (or more curved) than

any piece of the donut.

Algebra is the branch of mathematics concerning the study of the rules of operations and relations, and the constructions and concepts arising from them, including terms, polynomials, equations and algebraic structures. Together with geometry, analysis, topology, and number theory, algebra is one of the main branches of pure mathematics.

Combinatorial topology is today called algebraic topology. The origin of the name comes from the fact that that in combinatorial topology one cuts e.g. a surface into pieces, which are then put back together. The question whether it is possible to put the pieces back together in a significantly different way, is often reduced to a combinatorial question. The change of name for the field was actually initiated by the German mathematician Emmy Noether, as a consequence of her pioneering work in homology. A fairly precise date can be supplied in the internal notes of the French Bourbaki group. While topology was still combinatorial in 1942, it had become algebraic by 1944.

Algebraic K-theory is a general mathematical tool for classification of objects in several disciplines, invented by the French mathematician Alexander Grothendieck in the mid-fifties. Algebraic K-theory is the algebraic version of this general tool.

Some concepts

Exotic smooth spheres in seven dimensions

A sphere in seven dimensions is a generalization of the ordinary two-dimensional sphere. The circle is considered to be a one-dimensional sphere, the two-dimensional sphere is obtained from the circle by fixing two opposite points and spinning the circle around in the next dimension. Do the same with the two-dimensional sphere; one finger tip on each pole, north and south, and spin the sphere around in the next (non-visual) dimension. This gives the three-dimensional sphere. Continue up to seven dimensions.

Another description of spheres is via symmetry. The circle consists of all points in the plane of equal distance to one specified point, called the center. The sphere is defined similarly, as all points in space of given distance to the specified center. The seven-dimensional sphere is the set of all points in 8-dimensional space of fixed distance to a given centre. Exotic spheres are spheres (in any dimension) equipped with a differentiable structure (see next paragraph) different from the ordinary one.

Differentiable structures

It is possible to define a differentiable structure on almost any geometrical object, so let us use an ordinary sphere for illustration. To give a differentiable structure on a small part of the sphere, is more or less the same as drawing a map of that area. The mathematical problem is not to draw the map, but to make sure that all the different maps which cover the whole sphere fit together. Intuitively this does not seem to be particularly difficult, and in fact it is not, at least for an ordinary sphere. But if you try to do this for a 7-dimensional sphere, you have to make several choices on how to put the maps together. Milnor and Kervaire showed that it is 28 ways of doing this. One is rather straightforward, and can be done by a freshman at university level, the 27 others are more difficult to describe. Milnor called them exotic spheres.

Isolated hypersurface singularities

Walking along a road, you observe that there exist two different types of points along the road; the regular points, where you have no choice where to continue your trip, and the singular points, where you have to choose between two or more different possible roads. A singularity, as a general mathematical concept, is an abstract version of a crossroads. An isolated singularity means that there are no other singularities in a small neighbourhood of the point, and the word hypersurface is a technical description of how the singularity is defined.

Milnor number

There are in general many types of isolated singularities, even when we fix the dimension of the space. A very useful tool to characterize the singularity is to use the Milnor number. The singularity is put in the middle of a sphere of the appropriate dimension and we look at the intersection of the sphere and the space. This intersection will more or less look like a bouquet of spheres of one dimension less. The number of spheres is the Milnor number. A regular point has Milnor number 1, and the higher the Milnor number, the more complicated is the singularity.

The Hauptvermutung (German for main conjecture) of geometric topology is the conjecture that says that if we cover a surface by triangles under certain strict rules, and do this in two different ways, then we can always refine the triangulations, i.e. cut each triangle into several smaller triangles, such that the there is a common refinement of the two triangulations. Steinitz and Tietze originally formulated the conjecture in 1908, and it is proven to be wrong in high dimensions by Milnor.

A ***microbundle*** is a generalization of the concept of vector bundle. As for vector bundles, microbundles consist of two spaces, the total space and the base space, and a projection of one onto the other. Microbundles have an additional structure map in the opposite direction, from the base space to the total space, called the zero section. The fibres of the projection map, i.e. all points in the total space that are mapped to one specific point in the base space, look very much like vector spaces. The subtlety of microbundles, as of vector bundles in general, is to glue together the local pieces, the small local maps, to obtain a global object. It looks easy until one has to fit in the last piece, the one that ties everything together. That is the point where the concepts of vector bundle and microbundle expose their deep nature.

The degree two functor and the Milnor conjecture are concerned with a rather theoretical and not easily explained relation between to different expressions derived from "a field of characteristic different from 2". It relates Milnor K-theory to so called Galois cohomology. Milnor used his intuition to state the relation, but he was not able to prove it. The problem remained open for some two decades until Vladimir Voevodsky gave a proof of the relation around 1996.

Growth invariant of a group

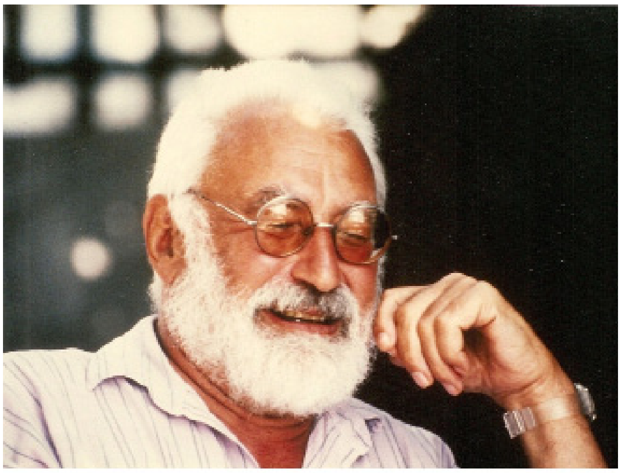
The most basic algebraic objects are called groups. Groups appear in many applications, in almost all disciplines of mathematics, in physics, chemistry and architecture, to mention a few. The Abel Prize in 2008 was awarded to John Thompson and Jacques Tits for their profound achievements in algebra and in particular for shaping modern group theory. One way of measuring the size of a group is to compute the growth of the group. We then start by a set of elements which generates the group. The growth measures the redundancy of this set. So a small group where we need a lot of different generators to describe the group has small growth. If the generators are by all means independent, the growth of the group is big. This is the case for the socalled free groups. The Abel Prize in 2009 was awarded to Mikhail Gromov, partly based on his work on the growth invariant of a group.

A dynamical system is a model for time evolution of a physical (or mathematical) system. Examples include the mathematical models that describe heat transfer in a rod, physical data of the atmosphere to predict the weather and time development of a population. At a certain time the system is in a state and the dynamical system gives a deterministic rule for how the system evolves to future states from the current state.

The Milnor–Thurston kneading theory is a theory which analyses the iterates of certain mappings of an interval into itself. The theory was developed by John Milnor and William Thurston in two widely circulated and influential Princeton preprints from 1977 that were revised in 1981 and finally published in 1988. The theory has a lot of interesting applications.

Some relevant names

Michel André Kervaire (1927-2007) was a French mathematician who collaborated with John Milnor in computing the number of exotic spheres in dimension greater than four. He was a professor at New York University's Courant Institute from 1959 to 1971, and then at the University of Geneva from 1971 until he retired in 1997.



John Coleman Moore (1927-) is an American mathematician whose most heavily cited paper is on Hopf algebras, co-authored with John Milnor.

William Paul Thurston (1946-) is an American mathematician. He is a pioneer in the field of low-dimensional topology. In 1982, he was awarded the Fields Medal for his contributions to the study of 3-manifolds. He is currently a professor of mathematics and computer science at Cornell University (since 2003). Thurston developed the Milnor-Thurston kneading Theory together with John Milnor in the 1978-80s.

Jules Henri Poincaré (1854-1912) was a French mathematician, theoretical physicist, engineer, and a philosopher of science. He is often described as The Last Universalist, since he excelled in all fields of mathematics, as it existed during his lifetime. Poincaré was responsible for formulating the Poincaré conjecture, one of the most famous and long-standing problems in mathematics. He is considered to be one of the founders of the field of topology.