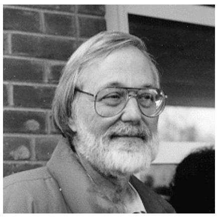


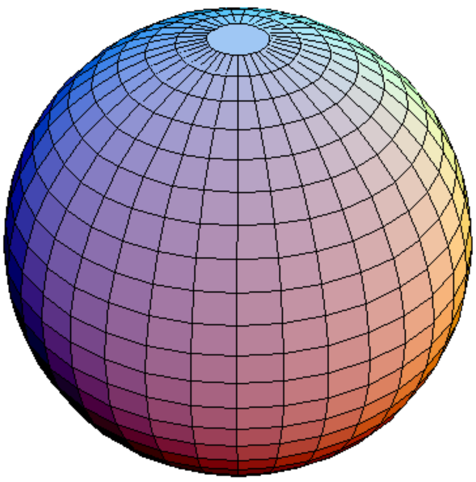
Abel Prize Laureate 2011
John Willard Milnor

Exotic Spheres in 7 Dimensions

The citation says: Milnor's discovery of exotic smooth spheres in seven dimensions was completely unexpected. It signaled the arrival of differential topology and an explosion of work by a generation of brilliant mathematicians; this explosion has lasted for decades and changed the landscape of mathematics. With Michel Kervaire, Milnor went on to give a complete inventory of all the distinct differentiable structures on spheres of all dimensions; in particular they showed that the 7-dimensional sphere carries exactly 28 distinct differentiable structures.

7-dimensional spheres

A sphere in seven dimensions is a generalization of the ordinary two-dimensioanl sphere. A circle is a one-dimensional sphere. Fixing two opposite points on the sphere and spinning the circle around in the next dimension we obtain a two-dimensional sphere. Do the same with the two-dimensional sphere, put your fingertips, one at the North pole and one at the South pole, and then



spin the circle around in the next dimension. This gives a (highly non-visual) three-dimensional sphere. Continue this procedure up to seven dimensions to get a seven-dimensional sphere.

A different point of view is to consider a sphere via its symmetries. A circle is the set of points in the plane of a given distance to a fixed point, the center of the circle. The two-dimensional sphere can be defined in a similiar way, as the set of points in space of a given distance from a chosen point. The seven-dimensional sphere is the set of points in an eight-dimensional space of given distance from the chosen center.

Differentiable structures

Defining a differentiable structure locally, i.e. on a small part of the sphere, is closely related to drawing maps of the sphere. There are different ways to do this. To define a differentiable structure on the whole sphere we need enogh maps to cover the sphere, and there must be rules for how to identifiy similiar points of the to maps on the

overlap between two maps.
Because of the curvature of the sphere, a map will be slightly incorrect towards the edges, in fact parallell lines will diverge. Corresponding lines on the next map will diverge in the opposite direction, and we must define a transition function between the two maps. The mathematical difficulty is not to draw the maps, but to make sure that they fit together, i.e. that the transition functions are consistent on the overlaps of the overlaps. In particular, one can imagine that fitting in the last map is not an easy task. Intuitively this doesn´t seem to be a very hard exercise, and on the two-dimensional sphere it is not. But try to to the same on a seven-dimensional sphere with overlapping seven-dimensional maps. Then it is more complicated.

Exotic spheres
To be able to fulfil this abstract task you have to make some choices. Milnor and Kervaire proved that it is 28 ways of doing this. One is more or less similar to how this is done in the two-dimensioanl case, but the other 27 are more complicated and neither of them have any analogy in two dimensions. Milnor called these complicated structures exotic spheres.

The number of different differentiable structures on spheres in low dimensions

dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
# structures	1	1	1	?	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

Status for dimension 4 is that it remains open. One does not know wether one has one, more than one or infinitely many smooth structures on the 4-sphere. The claim that there exist precisely one is known as “the smooth Poincaré conjecture in dimension 4”. The background for this is that when Michael Freedman in 1982 proved the Poincaré conjecture in dimension 4, i.e. that if a 4-manifold is homotopy-equivalent to a 4-sphere, then it is also homeomorphic to a 4-sphere, he left an open question for future investigations. If a 4-manifold is homotopy-equivalent to a 4-sphere, is it necessarily diffeomorphic to a 4-sphere? Milnor´s exotic spheres show that the smooth Poincaré conjecture is actually wrong in dimension 7.