Gauge Theory

Gauge theory is a mathematical theory introduced by Hermann Weyl in 1918, which originated in theoretical physics and Einstein's theory of general relativity. A key idea in Einstein's work is that laws of physics should be the same in all frames of reference. This is also the general idea of a gauge theory; to find connections that compare measurements taken at different points in a space and look for quantities that do not change. This physical interpretation was brought further by Yang and Mills in the fifties, in what is now called the Yang-Mills equations.

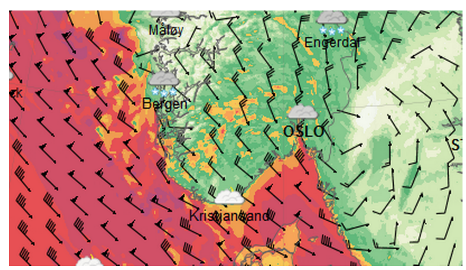
To reveal the secrets of theoretical physics, you have to work in a (at least) four-dimensional space (three spatial coordinates and one time-coordinate). A physical law should be the same wherever you are located in space-time, i.e. independent of the choice of frame of reference.

C. N. Yang (1922 -) and Robert Mills (1927 - 1999) at Stony Brook in 1999.

Karen Uhlenbeck attacked this problem from the mathematical point of view. After hearing a talk by Michael Atiyah in Chicago, Uhlenbeck became interested in gauge theory. She pioneered the study of Yang-Mills equations from a rigorous analytical point of view. Her work formed a base of all subsequent research in the area of gauge theory.

Gauge theory is usually considered to be a notion borrowed from physics, but the concept has also a pure mathematical life. A gauge can be considered to be a choice of a coordinate system, chosen in order to be able to measure various quantities. A gauge transform describes how to change coordinate systems, and a gauge theory models some physical or mathematical system which is most likely to be gauge invariant, i.e. it concerns quantities which are left unchanged under gauge transformations.

As an example we consider the circle bundle on a sphere S. In each point $x \in S$ on the sphere the space of directions can be identified with the standard circle S^1. To be able to measure a direction we have to pick an orientation and a reference direction. A good choice could be straight north as the reference direction and counterclockwise as the orientation. But any other choice could work as well. The circle of directions at the point x on the sphere is denoted by C_x. The collection $C_X = \{C_x\}_{x \in X}$ of all circles of directions in all points on the sphere is called the circle bundle. Notice that specifying directions in any point on the sphere is what we call a section of the bundle. If the sphere is the earth and the directions are wind directions, then a section s of the circle bundle gives a global wind direction map.



To be able to present a useful weather forecast we need to agree on the choice of reference system for the directions in the points on the sphere, i.e. choosing a gauge for the circle bundle. The numerical value of the wind direction or the more traditional notation like SSW is only useful for us when we agree on the reference system, but the wind itself is gauge invariant and independent of the choice of gauge.

An example of gauge invariance in physics is electromagnetism, modelled by Maxwell's equations. If $\mathbf{E}$ denotes the electric field, $\mathbf{B}$ the magnetic field, ϕ the electric potential and $\mathbf{A}$ the vector potential, we have the relations

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

Using the (purely mathmatical) fact that

$$\nabla \times \nabla = 0$$

and that

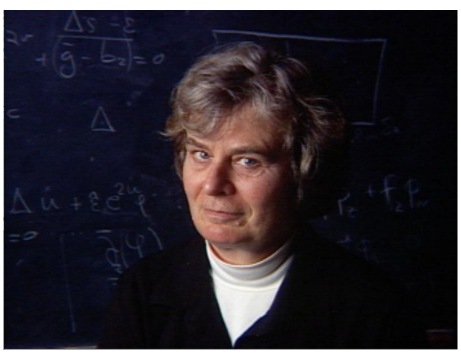
$$\frac{\partial}{\partial t}\nabla f = \nabla \frac{\partial f}{\partial t}$$

we see that substitution of

$$\mathbf{A} \mapsto \mathbf{A} + \nabla f$$

$$\phi \mapsto \phi - \frac{\partial f}{\partial t}$$

in a twice differentiable function f does not change the equations, i.e. Maxwell's equations are gauge invariant under the given gauge transformations.



Karen Uhlenbeck, Abel Prize Laureate 2019