

I'm Forever Blowing Bubbles ¹

**I'm forever blowing bubbles,
Pretty bubbles in the air.
They fly so high,
Nearly reach the sky,
Then like my dreams,
They fade and die.**

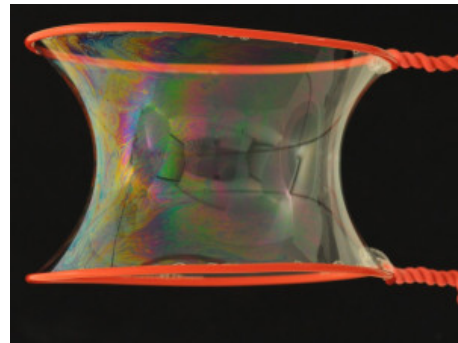
Soap bubbles are beautiful objects, perfectly shaped and with a marvellous play of colours, due to interference of light reflecting off the front and back surfaces of the soap film. Soap bubbles are beautiful objects in a mathematical setting as well, as they constitute examples of minimal surfaces. When the enclosed volume of air inside the bubble is fixed, the soap film will minimize the wall tension, pulling the bubble into the shape of the least surface enclosing a fixed volume, known for centuries to be a perfect sphere.



If we instead of blowing the bubble, dip a heavily deformed wire loop into a soap bubble solution, the soap film will form a disc with its boundary given by the wire loop and of minimal area. Unlike the sphere-shaped bubble, this film has equal pressure on each side, hence it is a surface with zero mean curvature, i.e. the average curvature along all directions is zero. Even if the soap film almost instantly is able to form

a minimal surface, computing the shape of the surface analytically is a rather complicated task.

Among curves connecting two points in space, we can always find a shortest path. The analogous statement is not true for surfaces when considering their area. The problem is that in order to reduce the area of a surface, a consequence could be that the surface is shrunk to a curve, which of course does not count as a minimal surface. An example of this is the minimal tubular surface connecting two parallel circles.



Source: Soapbubble.dk

If the distance between the circles is small compared to their radius, the minimal surface looks like a slightly concave cylinder known as a catenoid. When pulling the circles apart the cylinder will shrink in the region between the circles. At a certain point, the middle part of the curved cylinder will collapse along the line connecting the centres of the two parallel circles. When pulling the circles further apart there is no tubular minimal surface connecting them.

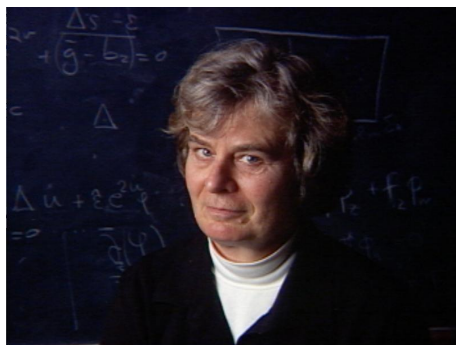
In 1968 Karen Uhlenbeck received her Ph.D. from Brandeis with the thesis "The Calculus of Variations and Global Analysis". Her supervisor was Richard Palais, who a few years earlier and together with Stephen Smale, had introduced the so-called Palais-Smale Condition C. This condition gives a criterion for the existence of minimizers for functionals on mapping spaces. "Minimizers for functionals on mapping spaces" is a more general phrase than "find the surface of least area", but Condition C can also be applied to the minimal surface problem. However, in that case it fails. Motivated by the general non-existence of minimal surfaces, Uhlenbeck wanted to

¹*I'm Forever Blowing Bubbles* is an American song from the Broadway musical *The Passing Show of 1918*. It was released in 1918, the same year as Hermann Weyl introduced the notion of a gauge. In addition to the obvious connection to minimal surfaces, the lyrics may have some associations to mathematical research in general. The song is also adapted as the club anthem of West Ham United, a London-based football club.



understand what happens when Condition C is violated. In a paper co-authored with Jonathan Sacks, she describes in detail the situation where you cannot rely on the conclusion of Condition C. They construct a sequence of mappings of a sphere into the target space which satisfies Condition C, but in such a way that their limit does not. Outside of a finite set of singular points everything works well, but near the singularities the so-called bubbling phenomenon appears. The area of the limit surface is strictly less than the limit of the areas of the surfaces in the sequence. The difference is concentrated in a finite set of isolated points, being the limit of "bubbles" in the sequence of surfaces.

The ideas and the methods of this revolutionary paper has since it was published become a successful mathematical tool. In particular, the bubbling phenomenon has had great influence as a method for solving problems in various parts of mathematics.



Karen Uhlenbeck, Abel Prize Laureate 2019