The first **Abel Prize** has been awarded to Jean-Pierre Serre, one of the great mathematicians of our time. Serre is an Emeritus Professor at the Collège de France in Paris. He has made profound contributions to the progress of mathematics for over half a century, and is continuing to do so. Serre has had a major influence in shaping the modern form of many parts of Mathematics, including

- *Topology*, which studies curves, surfaces, and more general geometric spaces and treats the question: What remains the same in the geometry of these spaces even when length is distorted? How can we tell whether one geometric object is essentially different from another in the sense that even assuming that both objects are as flexible, as pliable, as possible, no clever distortion of them can make them look like each other. For example, how can we tell that the two trefoil knots

are essentially different? Imagine them as made out of string. No fidgeting with them (without cutting) can make them look identical!

- *Number Theory*, which studies the basic properties of numbers, prime numbers, factorisation, algebraic numbers, questions of placement of numbers (e.g., Catalan's problem: "the only consecutive perfect powers are $2^3=8$ and $3^2=9$) questions of estimation (e.g., for very large $x$ there are roughly $\frac{x}{\log x}$ prime numbers which are less than $x$) and solutions of polynomial equations (e.g., Fermat's Last Theorem).

- *Algebraic Geometry*, which enormously strengthens the link between algebra and geometry that is already evident in the analytic geometry of high school mathematics, studying conics and ellipsoids. It is the branch of mathematics that asks these basic questions, among others: What is the geometry of solutions of systems of polynomial equations? Given information directly about that geometry, what does that tell us about the underlying algebra? How many degrees of freedom do we have in realising a specific topological space as the space of solutions of systems of polynomial equations?

Many of the fundamental theorems in these disciplines are due to Serre. And these are only some of the areas to which Serre has made contributions. We will take a closer look at only two topics, Topology and Number Theory. The modern era of Topology was begun with the work of Henri Poincaré in his series of articles *Analysis Situs* and we shall give some hints of the link between that origin and Serre's extraordinary work in algebraic topology. The modern era of Algebra and Number Theory may be said to begin with the work of Gauss, Lagrange, Galois, and Abel. Here again we will briefly evoke the connection between these origins and the one of Serre's contributions to Number Theory.

**Topology**: Serre developed revolutionary algebraic methods in Algebraic Topology. To get a sense of the type of mathematical question that is undertaken by algebraic topology, imagine that you have been given a geometric space $X$ and want to understand its qualitative features, and more precisely only those features that do not change even if you allow your geometric space to be flexible, and not rigid. For a simple example, let us take $X$ to be the surface of an inner-tube. This object, being made of rubber, is very likely to be subjected to stretches and scrunches, and we want to focus on those features of that surface that are unchanged by such modifications of its shape. No measurement of length on our surface, for example, ranks as such a legitimate "qualitative feature," for distortions of the rubber can change any such measurement. The great mathematician Henri Poincaré discovered, about a century ago, an interesting "qualitative feature" of such a surface by asking the following type of question: in how many ways can you draw a closed curve on that surface, where you deem two "drawings of closed curves on the surface" essentially different only if each of these drawings cannot be continuously modified to become the other.

In classifying, and studying, the full collection of essentially different closed curves in the geometric space $X$ mathematicians found a source of, and vocabulary for, the type of "qualitative features" they sought. The answer to Poincaré`s question: "How many essentially different closed curves are there in $X$?" is sometimes enough to distinguish between spaces (for example, if your space $X$ is the surface of a ball, there is only one, while if $X$ is the surface of a doughnut, as the figures above illustrate, there are many of them).

Where else can one find "qualitative features" which are perhaps even more sensitive tools to distinguish, and understand, geometric spaces $X$? (Mathematicians refer to these "qualitative features of $X$" as "homotopy invariants of $X$.") A natural place to look for more and finer "qualitative features," or "homotopy invariants," is in higher dimensional versions of the idea of Poincaré. That is, instead of considering only the tool used by Poincaré, i.e., closed curves, in the space $X$, we might focus on essentially different higher dimensional spaces of particular types contained in the space $X$. But what higher dimensional spaces are particularly good to use as a source of qualitative features?

Now the n-dimensional sphere is the space of all points of distance 1 from the origin in n+1 dimensional Euclidean space: the one-dimensional sphere is the circle, the two-dimensional sphere is the surface of a ball, etc. Moreover, a closed curve in $X$ is gotten by placing the one-dimensional sphere, the circle, in $X$ (via, as mathematicians say, a continuous mapping).

This led mathematicians to consider, for each dimension *n*=1,2,3,… the set of essentially different continuous mappings of the n-dimensional sphere to *X*. These sets (appropriately defined) are referred to as the n-th homotopy groups of *X* (for *n*= 1,2,3,…) and they rank as a series of great, mysterious "qualitative features" of the geometric space *X*. We have used the mathematical term group to signal that there is an intricate algebraic structure to the set of these essentially different mappings: one of the great surprises of the subject is that the search for "qualitative features" leads ineluctably to algebraic objects.

For homotopy groups to be useful as a tool we had better be able to compute them for basic spaces *X*, and specifically for the spheres themselves. For every pair, then, of natural numbers *n* and *m*, the core problem is to compute the number of, and more exactingly the structure of, essentially different mappings of the *n*-dimensional sphere to the *n*-dimensional sphere. There is always one, uninteresting, continuous mapping: the mapping that projects the *n*-dimensional sphere to a single point on the *m*-dimensional sphere. But for many $n \geq m$ there are also *interesting* mappings, intricate and beautiful, and with connections to algebra and, perhaps more surprisingly, number theory. Except when $n = m$ or $n = 2m-1$, the number of essentially different mappings from the *n*-dimensional sphere to the *m*-dimensional sphere is finite. Detailed knowledge of them is the key to attacking a wide range of topological problems.

Serre developed an algebraic machine that produced precise answers to many such questions, and that launched an era of algebraic topological investigation; to give a sense of the intricacy and precision here, let us just quote a few instances: there are only two essentially different continuous mappings of the 5-dimensional sphere to the 3-dimensional sphere, there are 12 essentially different mappings from the 6-dimensional sphere to the 3-dimensional sphere, there are 4 essentially different mappings from the 9-dimensional sphere to the 4-dimensional sphere.

**Number Theory:** For the past four decades Serre's magnificent work in number theory, and his vision, have brought that subject to its current glory. His work has been vital in setting the stage for many of the most celebrated recent breakthroughs, including the work of Wiles in Fermat's Last Theorem. Serre's contribution here is so vast, that it hardly is possible to give a sense of its immensity, but let us try to explain a bit of background to appreciate just one of Serre's fundamental results.

The quadratic formula taught in high school "solves" all quadratic polynomial equations in one variable in terms of square roots of known things. Sixteenth century Italian algebraists used cube roots, and fourth roots to express the solutions of polynomial equations of degrees three and four. The great Norwegian mathematician Niels Henrik Abel showed that this simple tool ("extracting roots") was not adequate to solve all polynomial equations of higher degree.

Nevertheless the idea of "extracting roots" remains a formidable, if not universally successful, device to analyse *algebraic numbers*, that is, numbers that occur as solutions of polynomial equations with integer coefficients. Here is an example of such a polynomial equation, chosen at random:

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

This equation has (as it turns out) six solutions; all of these solutions, therefore, are examples of algebraic numbers.

In the early 1800's the mathematician Carl Friedrich Gauss concentrated upon, among many other things, a certain core problem regarding roots. Specifically, Gauss offered a deep analysis of those algebraic numbers that are roots of the number 1: if $z$ is a complex number whose cube is 1, then $z$ is called a cube root of 1, and similarly, if $z$ is a complex number, which when raised to the $n$-th power gives 1,

$$z^n = z\,z\,z \ldots z = 1$$

then $z$ is called an $n$-th root of 1. For example, the collection of all complex numbers $z$ which are seventh roots of 1 consist of 1 itself, and six other complex numbers which, expressed in the language of elementary trigonometry, are:

cos (2p/7) + i sin (2p/7)
cos (4p/7) + i sin (4p/7)
cos (6p/7) + i sin (6p/7)
cos (6p/7) - i sin (6p/7)
cos (4p/7) - i sin (4p/7)
cos (2p/7) - i sin (2p/7)

(These six numbers are also precisely the solutions of the sixth degree equation given above, as you can check using high school trigonometry).

If you situate all seventh roots of 1 in the complex plane they form the vertices of a regular heptagon (a seven-sided polygon) and, in fact, nth roots of 1 play an important role in the history of geometry, and in particular, in Euclidean constructions.

One of Gauss' celebrated results about roots of 1 provides a guarantee that they are a rich source of algebraic numbers held tightly together by internal symmetries. What is meant by "rich" is that, as Gauss showed, there is no redundancy in the above list of the six different 7-th roots of 1. That is, no positive whole number multiple of any root on that list can be given as a sum of whole number multiples of the other items on the list. He proved that for each $n$=3,4,…, the $n$-th roots of 1 are similarly a "rich source."

One of Serre's many fundamental theorems guarantees a vastly richer reserve of algebraic numbers (which are held tightly together by internal symmetries) coming to us via a study of mathematical objects called elliptic curves.

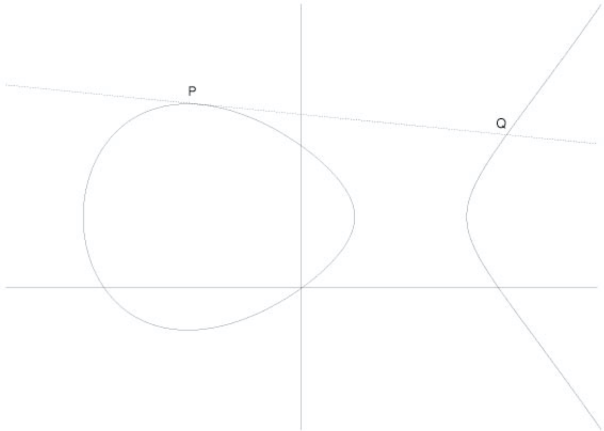
*Elliptic curves* emerged from the study of elliptic integrals, which themselves arose as expressions for the lengths of arcs of ellipses and of closely related curves in the plane. It was the idea of Gauss, Abel, and Jacobi in the early nineteenth century to *invert* certain of these arc-length expressions giving functions that generalise the classical functions of trigonometry (sine, cosine). Elliptic functions satisfy an *addition law* analogous to the trigonometric addition law of sine and cosine. Moreover, instead of the simple quadratic equation ($\sin^2 x + \cos^2 x = 1$) satisfied by sine and cosine, the elliptic functions satisfy a cubic polynomial relation and their values trace out plane cubic curves. These are *elliptic curves*.

Elliptic curves of particular relevance to number theory are smooth cubic curves in the $(x,y)$-plane with whole number coefficients such as

$$E: \quad y^2 + y = x^3 - x$$

For each such elliptic curve $E$ we will see how to organise a certain collection of algebraic numbers obtained by studying $E$; these algebraic numbers are the coordinates of particular points on the elliptic curve. Mathematicians call the points we will be discussing "torsion points on $E$ of odd order." In the study of the elliptic curve $E$, torsion points play a role similar to that played by roots of 1 in the study of the complex numbers.

Here is a clean geometric description of this collection of particular points on an elliptic curve $E$: Start with a point $P$ on $E$ with coordinates $P = (a,b)$. Draw the tangent to the curve $E$ at $P$; this tangent line will intersect the curve in exactly one other point $Q$ (except for the eight instances when that tangent line hugs that curve a bit too much, for in these cases the "other point" $Q$ is to be thought of as $P$ itself!). Now do the same with $Q$; draw the tangent to the curve at $Q$; this tangent line will intersect the curve in exactly one other point $R$. Now do the same with R, and keep going in this manner. Either you never return to the point $P$ no matter how many times you iterate this process of drawing tangent lines and passing to the other intersection point, or else you do return to $P$. If you do return to $P$, then the coordinates of $P$ are, of necessity, algebraic numbers. Such points $P$ are, as signalled above, referred to as torsion points of odd order.




Serre´s vast generalisation of the result of Gauss described above has the effect of guaranteeing that for any elliptic curve, this collection of algebraic numbers is also as rich as one might optimistically hope for and held tightly together by an intricate system of symmetries.

To give a hint of what is meant by this, we shall make use of, but not define, a number referred to by mathematicians as the *order* of a torsion point of an elliptic curve. Consider the case of the elliptic curve

$$E: \quad y^2 + y = x^3 - x$$

For any prime number $p$ there are precisely $p^2-1$ torsion points on this curve of order $p$, all of them, to be sure, having algebraic numbers as $(x,y)$-coordinates. Serres result guarantees that for any prime number $p$, there are as many as $(p^2-1)(p^2-p)$ different permutations of this set of torsion points that preserve their underlying algebraic structure.

Serre's theorem offers information about *all* elliptic curves given by cubic equations whose coefficients are algebraic numbers, and not just our example. A consequence of it is that for any such elliptic curve either the above state of affairs holds for all but a finite number of primes $p$, or else the elliptic curve itself is of an exceedingly special kind (*is "of complex multiplication"*).

Serre's work was at the beginning of the glorious modern epoch in the subject, where this deep arithmetic behind elliptic curves connects with more classical subjects such as modular forms, and where Serre himself has done inspiring and profound work.