THE ABEL PRIZE 2020

Random walk, a good strategy to find the treat, –for the dog as well as the mathematician

If you hide some treats in your garden, and you ask your dog to find it, she will immediately start to seek. The trajectory along which she is sniffing seems to be quite random. The dog is apparently not a fan of a systematic way of seeking. But nevertheless, after a short period of time the dog finds the reward. The dog's instinct tells her to use a random walk technique, i.e. the direction of her search constantly changes in a random way.

In mathematics the dog's behaviour is encoded in the concept of random walk. A random walk is a mathematical object that describes a path consisting of a succession of random steps in some mathematical space. There are numerous examples of physical systems that are modelled by random walks, behaviour of gas molecules, stock markets, the statistical properties of genetic drift and neuron firing in the brain, to mention some. But random walks can also be seen as a tool to explore a mathematical object, in the same way as the dog tries to understand the garden. Harry Furstenberg and Gregory Margulis are not using random walks to find treats in the garden. They do the random walk on graphs or on groups in order to reveal the secrets of these objects.

A popular family of mathematical objects is the so-called Lie groups, named after the Norwegian mathematician Sophus Lie (1842–1899). Lie groups are objects that describes the symmetries of geometrical objects, such as rotational symmetry in three dimensional space. Sophus Lie was inspired by earlier work of Abel and Galois on solutions of algebraic equations. Abel's proof of the non-solvability of the fifth degree equation in radicals, and Galois' ground-breaking theory for connecting solutions of polynomial equations to certain automorphism groups of field extensions, are both brilliant examples of how one can understand the details by extending the horizon. Lie's idea was to introduce a similar way of studying symmetries of differential equations. It has ever since been an important task to understand the structure of such groups, in order to approach the solution of the underlying differential equations.

Furstenberg and Margulis have through their invention of concepts and proof of theorems made significant contributions to our understanding of Lie groups. In general, a Lie group is infinite and non-compact, i.e. regardless of how we consider the group, it will possess some unboundedness. Random walk techniques are well suited to capture the nature of the unbounded.

If the trajectory of the dog is ergodic, the dog will in the long run get close to the treat. In fact, if we draw a circle around the treat, of arbitrary small radius, she will after some time be sniffing inside the circle, and probably discover the treat. This is an example of recurrence. Recurrence of a dynamical system stems back to work of Henri Poincarè towards the end of the nineteenth century. He proved that under certain conditions a dynamical system (a system that develops in time) will return, or at least almost return, to any point in the ambient space. Using random walk technique, we can connect the size of the group to the question of recurrence. If the group is ”too large”, it is likely that no recurrence will occur during the random walk, and vice versa.

The mathematical heritage of Harry Furstenberg and Gregory Margulis contains many inventions based on ergodic theory, recurrence, Lie groups and random walks. Furstenberg introduced Furstenberg boundaries and disjointness, Margulis came up with concept of superrigidity and the normal subgroup theorem. Margulis also gave a proof for the Oppenheim conjecture, concerning integral almost-solution to quadratic equations in three variables and Furstenberg confirmed Endre Szemerédi's theorem on the existence of arithmetic progressions of any length, using ergodic theory. The two last examples are nice examples of how the two laureates demonstrate the ubiquity of probabilistic methods and the effectiveness of crossing boundaries between separate mathematical disciplines, as pointed out in the citation of the Abel comittee.

The mathematical garden has many hidden treats, the work of this year's Abel Prize laureates suggests that random walk could be a good strategy to reveal some of them.