THE ABEL PRIZE 2020

Recurrence of random walks in $\mathbb{Z}$ and $\mathbb{Z}^2$, but not in $\mathbb{Z}^3$

Suppose that we at a certain time were able to localize every gas molecule in a closed container, writing up their position and velocity, i.e. decide the state of the gas. Just after this golden moment the gas molecules will continue to move around in the container, constantly collide with each other and the walls in a rather chaotic manner. It seems a bit paradoxical that the gas once more should reach the same state as we observed earlier, but according to Henri Poincarè's recurrence theorem from 1890 this is exactly what happens. You may compare it with the fact that if you continue to write up letters randomly, at some point you will by accident have written a strict correct version of the famous Henrik Ibsen play A Doll's House. It may take some time, but sooner or later it will happen.

Various results about recurrence have been presented since Poincarè's discussion back in 1890. The mathematical foundation of the concept was laid by Birkhoff in 1930, in his proof of the ergodicity theorem. The theorem says that for a measure-preserving transformation on a finite dynamical system almost every starting point of a process will be repeatedly reached.

To illustrate the recurrence phenomenon we can study random walks in $\mathbb{Z}^n$ for $n = 1, 2, 3$, with uniform probability distribution.

Consider a uniform distributed random walk along $\mathbb{Z}$. Located at x, in the next step we will reach $x+1$ with probability $\frac{1}{2}$ and $x-1$ with the same probability. Let $M \in \mathbb{Z}$, for simplicity we assume $M > 0$. For any $0 < x < M$ we ask the question: Which number do we reach first, 0 or M? Let $m(x)$ denote the probability that we reach M first. What happens in the next step? With probability $\frac{1}{2}$ we move to $x+1$, where the probability of reaching M first is $m(x+1)$. With the same probability we move to $x-1$, where the corresponding probability is $m(x-1)$. This sets up a recursion

$$\tfrac{1}{2}m(x-1) + \tfrac{1}{2}m(x+1) = m(x)$$

with boundary condition $m(0) = 0$ and $m(M) = 1$. The solution of this difference equation is $m(x) = \frac{x}{M}$. The probability of reaching 0 first is then $1 - \frac{x}{M}$. Suppose we do not reach 0 at all. That is equivalent to the fact that we reach any positive number M before we reach 0, i.e. $m(x) = \frac{x}{M} = 1$ for all $M \in \mathbb{Z}$ and any start value x. This is of course impossible, which means that we for sure will return to 0.

In fact we can compute the probability that the random walker will be back after $2n$ steps (it has to be an even number). Using the Pascal triangle we see that the probability of returning after $2n$ steps is

$$\frac{\binom{2n}{n}}{2^{2n}}$$

This number can be approximated for large values of n by Sterling's formula. Sterling's formula says that

$n! \sim n^n e^{-n}\sqrt{2\pi n}$, thus we get

$$\frac{\binom{2n}{n}}{2^{2n}} \simeq \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{2^{2n}(n^n e^{-n}\sqrt{2\pi n})^2} = \frac{1}{\sqrt{\pi n}}$$

Let p be the probability of returning to 0. Then the probability of returning exactly n times is $p^{n-1}(1-p)$. The expectation of this distribution is

$$E = \sum_{n=1}^{\infty} np^{n-1}(1-p) = \frac{1}{1-p}$$

The expectation of number of returns to 0 can be expressed as the sum of the expectations of only one return after $2n$ steps;

$$E = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}}$$

which diverges and it follows that $1-p=0$, i.e. we will for sure return to 0.
We can use the same formalism to show recurrence in $\mathbb{Z}^2$. In that case we have to return to 0 in two directions, i.e. the probability is given by

$$\frac{1}{\sqrt{\pi n}} \cdot \frac{1}{\sqrt{\pi n}} = \frac{1}{\pi n}$$

Again we have that

$$E = \sum_{n=1}^{\infty} \frac{1}{\pi n}$$

diverges and $1-p=0$.
If we continue to increase the rank, i.e. looking at $\mathbb{Z}^3$, we notice that the series

$$\sum_{n=1}^{\infty} \frac{1}{(\sqrt{\pi n})^3}$$

converge, and we get $p \neq 1$, which means that we can not sure that we will return to 0. The probability of returning can be computed to approximately $p = 0.3405$.