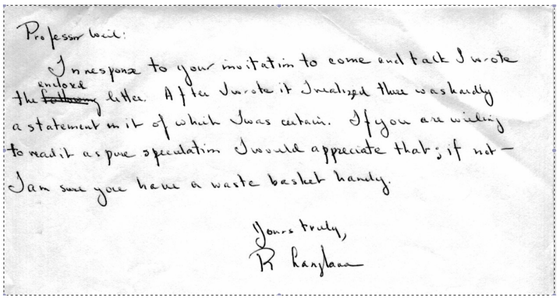
17 handwritten pages that shaped a whole area of mathematical research



Professor Weil:

In response to your invitation to come and talk I wrote the ~~following~~ enclosed letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation I would appreciate that; if not — I am sure you have a waste basket handy.

Yours truly,
R Langlands

"In response to your invitation to come and talk I wrote the enclosed letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation I would appreciate that; if not - I am sure you have a waste basket handy."

This is the text of the cover page of Robert P. Langlands' 17-page handwritten letter to André Weil in January 1967. André Weil, at that time in his early 60s, was one of the most influential individuals in mathematics during the 20th century, particularly in algebraic geometry and number theory. Robert Langlands was 30 years younger, a promising mathematician, but still in an early stage of his career. Weil did not respond to the letter, but he had it typed, and this typed version circulated widely among mathematicians. The content of the letter would soon be known as the "Langlands conjectures".

A famous result in number theory states that an odd prime number p can be written as a sum of two squares if and only if the prime number has remainder 1 when divided by 4. In mathematical terms this is phrased by saying that p is congruent to 1 modulo 4. The result was formulated by Fermat already in 1640, and proved by Euler a hundred years later. The integers 5, 13, 17 and 29 are the first primes that are congruent to 1 modulo 4, and corresponding decompositions into a sum of two squares are $5 = 1^2+2^2$,$13 = 2^2+3^2$, $17 = 1^2+4^2$ and $29 = 2^2+5^2$. On the other hand there are no ways of writing 3, 7, 11 or 19 as sums of two squares, since their remainder is 3 when dividing by 4. This result is an example of a **reciprocity law**, as it expresses a subtle arithmetic property of a prime number p, in this case, representability as a sum of two squares, in terms of a congruence condition on p.

Representation of a prime p as a sum of two squares is equivalent to a factorization of p as a **Gaussian integer**. A Gaussian integer is a complex number whose real and imaginary parts are both integers. The prime number p factors as a Gaussian integer if and only if there exist integers m and n such that

$$p = (m + in)(m - in)$$

Here i stands for the imaginary unit, with the defining property $i^2 = -1$. A simple computation shows that the existence of such a splitting is equivalent to $p = m^2 + n^2$, i.e. p has a representation as a sum of two squares.

The splitting of a prime p as a Gaussian integer is closely related to what is called the **Galois theory** of the **Gaussian rationals**. The set of Gaussian rationals is the rational counterpart of the Gaussian integers, i.e. complex numbers whose real and imaginary parts are both rational numbers. For each prime number p there exists a particular element in the so-called **Galois group** of the Gaussian rationals, the **Frobenius automorphism**, whose order is crucial for deciding whether p can be represented as a sum of two squares. The order of an automorphism is the least number m such that the m-th iteration of the automorphism is the identity. The Frobenius automorphism is given by raising a number to i ts p-th power. The p-th power of the imaginary unit i for an odd prime p is either ii or $-i$, depending on whether the remainder of p when divided by 4 is 1 or 3. Thus the Frobenius automorphism gives the bridge between the arithmetic property and the congruence property of the prime p.

The content of Langlands' letter to Weil suggests a far-reaching generalization of the result on representations of prime numbers as sums of two squares. It seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles.

A major task in number theory is to find integer solutions to equations with integer coefficients. It is rather obvious that if there exists an integer solution, the equation can also be solved modulo any power p^k of a prime p, the solution being the remainder of the integer solution modulo p^k. Kurt Hensel reformulated in 1897 this statement by introducing the so-called p-adic numbers, which reduced the set of statements for all powers of p to just one statement about the p-adic numbers. The set of p-adic numbers is an example of a **local field**. A famous theorem, formulated by Hermann Minkowski and later generalized by Helmut Hasse gives a positive answer to the question of reversing the order of the statement, featured as the local-global principle; does the existence of a solution to an equation in the p-adic numbers ensure the existence of an integer solution? The Hasse-Minkowski theorem tells us that this is true for a quadratic equation. But it is not a general fact. A famous counterexample is the Fermat equation $x^n + y^n = z^n$, for $n \geq 3$ proven to have no integer solutions by Abel Prize Laureate Andrew Wiles in 1995. But it was known already in 1909 that the equation has solutions in the p-adic numbers for any prime p.

Although the local-global principle is not valid in general, p-adic numbers play a prominent role in number theory. An even more prominent role is played by the collection of all p-adic numbers together with the real numbers, encoded in an algebraic object called the **adeles**.

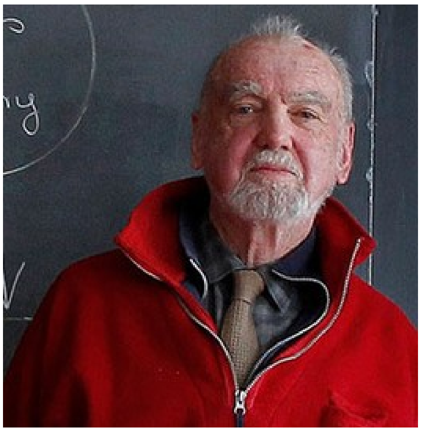
At the beginning of January 1967, Langlands and Weil met coincidentally in a corridor of the Institute for Defense Analysis in Princeton, both having arrived early for a lecture by Shiing-Shen Chern. Not knowing quite how to start a conversation, Langlands began to describe his reflections of the connections between number theory and automorphic forms. Weil, using "a well-known stratagem to escape politely from importunate individuals" (in Langlands own words), suggested that the young colleague could send him a letter describing his thoughts.

Automorphic forms were introduced by Henri Poincar'e in the 1880s as part of his doctoral thesis. As a first approach one can view an automorphic form as a function of the upper half complex plane, subject to a certain periodicity. In Langlands' visionary work he uses an extended definition of an automorphic form, as a certain representations of the adeles, still subject to some periodicity.

Remember that the Langlands correspondence seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles. Also remember, that Langlands points out a program of research, and not necessarily a list of proved theorems.

The most famous example of this correspondence is the modularity theorem, for which Andrew Wiles was awarded the Abel Prize in 2016. The Taniyama-Shimura-Weil conjecture predicts a close connection between the number of solutions to a type of equations, called elliptic curves, and a particular type of automorphic forms, called modular forms. In this example the representation theory of the Galois group of a maximal extension of the rational numbers produces a sequence of numbers encoding the number of solutions to an elliptic curve modulo various prime numbers p. The Langlands correspondence relates this sequence to a sequence of numbers characterizing an automorphic form over the adeles. Thus the Taniyama-Shimura-Weil conjecture, and consequently Fermats last theorem, follows as a special case of the Langlands correspondence.

This is just one example of how Langlands' ideas have influenced different areas of mathematics. It is really no exaggeration to say that his 17 handwritten pages has shaped a whole area of mathematical research.



Robert P. Langlands