

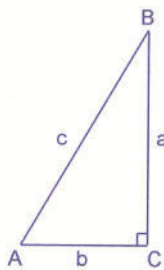
## Abel Prize Laureate 2011 John Willard Milnor

### *Topology vs. Geometry*

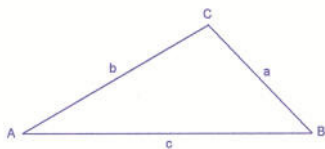
Classification of various objects is an important part of mathematical research. How many different triangles can one construct, and what should be the criteria for two triangles to be equivalent? This type of questions can be asked in almost any part of mathematics, and of course outside of mathematics. We classify human beings by their sex or age. The weather for tomorrow is classified by expected temperature, pressure and amount of clouds. The basis for any classification is to decide which objects should be considered to be equivalent. The difference between topology and geometry is of this type, the two areas of research have different criteria for equivalence between objects.

#### Classification of triangles

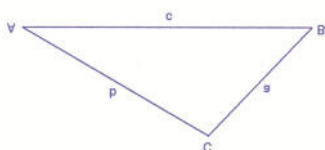
Here is a triangle:



Here is another triangle:



And this is also a triangle.



Three quite different objects, but all triangles. The reason is obvious, all three objects satisfy the

criteria of being triangles, the boundary is piecewise linear and consists of three edges. Every object that fulfill this requirement is called a triangle. This is a reasonable way of classifying objects with straight edges, so-called polygons, we just count the number of edges. There are some common properties of all polygons.

#### Property no. 1 for polygons:

*You have to see the whole figure to decide the number of edges.*

If you only see a small part of the interior of a polygon you will not be able to decide whether the polygon has three or four or more edges. Being a polygon is what we call a global property. The opposite is a local property. The size of an angle is a local property. You only need to glimpse a tiny neighbourhood around the vertex of the angle to decide the size of the angle.

#### Property no. 2 for polygons:

*Size and shape (as long as they are polygons) are irrelevant.*

There are an infinite number of different triangles. The point is that we do not care about lengths or sizes of the angles, we just count them. Being a triangle is not a geometric property, since geometry involves measuring. The property of being a triangle is more like a topological property. The word topology is composed of the Greek terms “topos” which means place and “logos” which



means word, speech or reason.

As a mathematical branch, topology has its origin in the work of Henri Poincaré towards the end of the nineteenth century, but can also be linked to the great Gauss. In what is now called the Gauss-Bonnet theorem, Gauss, and later Bonnet, showed that the sum of the curvature for every point on a balloon is independent of the local shape of the balloon. If we deform the balloon by pressing a finger tip onto the surface, the total curvature remains unchanged. In some points the curvature will increase, in some points it will decrease, but as told to us by Gauss, added up the changes cancel out each other. Total curvature is therefore a global property, and in fact only dependent on the topological class of the balloon.

### Congruence of triangles

There is another familiar property of triangles, called congruence. Two triangles are congruent if the angles and the lengths of the edges are pairwise equal. This gives us a new way of classifying triangles. In the former classification, all triangles were equivalent, there exists, so to speak, only one triangle.

In the new classification the requirement for being equivalent is much stronger. Congruence is a very strict property, as it is reflected in the huge number of equivalence classes of congruent triangles. Each of the three vertices is given by a pair of coordinates, adding up to a 6-dimensional parameter space. In this space there is a 5-dimensional subspace where the three points are rectilinear (two points are chosen freely and the third has to stay on the line through the other two). We avoid this subspace since it parametrizes degenerated triangles, which are no more true triangles. Permuting the vertices gives equivalent triangles, in general 6 symmetries. Of course two triangles should be equivalent if they are translates. There is a 2-dimensional family of translates in the plane, cutting the parameter space down to four dimensions. Finally there is a 1-dimensional family of rotations, reducing the dimension once more. In the end we get a 3-dimensional parameter space of triangles, with a 2-dimensional subspace of

degenerated triangles, and a finite symmetry-based reduction. The three parameters can be viewed as the lengths of the three edges, and this triple is known to decide the congruence class of the triangle.

### Similar triangles

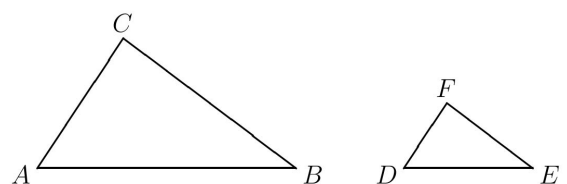
In between the coarse classification where we just count the edges, and the finer classification of congruent triangles, there is an intermediate one, given by similarity. Similar triangles is almost the same as congruence, but now the size of the triangles is unimportant. Thus the parameter space of similar triangles is one dimension less than the congruence space, i.e. 2-dimensional, still with the finite symmetry-reduction. If we denote by  $a$ ,  $b$  and  $c$  the lengths of the edges, the 2-dimensional space of similar triangles has parameters  $a/c$  and  $b/c$ .

It is important to remember that whatever method of classification, the triangles remain the same. The difference is which of them we consider to be equivalent.

### What about Milnor?

When Milnor states that 7-dimensional spheres are homeomorphic, but not diffeomorphic, the reasoning is analog to our treatment of triangles. Being homeomorphic is less strict than being diffeomorphic, just as being similar is less strict than being congruent.

Before Milnor's result many people believed that for spheres the two concepts were equivalent. Milnor showed that this is not the case, being diffeomorphic *is* more strict than being homeomorphic.



To formlike trekanter