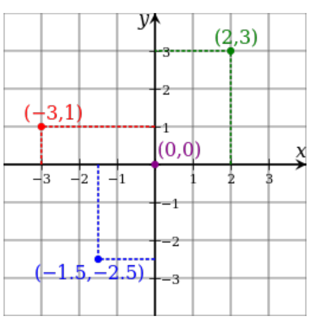
Yves Meyer, Abel Prize Laureate 2017

ORTHOGONAL FUNCTION SYSTEMS

Orthogonal function systems play an important role in many areas of mathematics. The basic idea is to try to find a limited number of "basis functions", which can serve as building blocks for all other functions. The basis functions should have certain nice properties, in order to simplify the construction.

About 400 years ago the philosopher and mathematician René Descartes introduced the concept of a Cartesian coordinate system, which gives a simple way to describe the location of a point. What we need is a reference system. The reference system is determined by three essential characteristics; the location of the origin, the direction of the axes, and a scale. In addition we require the axes to be mutually orthogonal. Related to this system the coordinates of a point are unique, and easily computed.



This simple idea can be generalized to more abstract applications. Points in space are substituted by functions. The coordinate axes are replaced by certain basis functions and the scalar product which determines the coordinates is replaced by a more general **inner product**. As a coordinate of a point is interpreted as the component of the point in a given direction, the coordinate of a function along a basis function will determine the "component" of the function in that direction. If the basis functions are orthogonal, i.e. their mutual inner products vanish, the system of basis functions is called an **orthogonal function system**. If in addition the basis functions are normalized, i.e. the inner product of a function with itself is 1, we say that the system is **orthonormal**.

The usual notation for an inner product of two functions f and g is $\langle f, g\rangle$. An inner product has many nice properties; one important fact is that the inner product of a function with itself is non-negative. Another fact is that there is only one function with vanishing inner product with itself: the constant function 0.

There are many examples of orthogonal function systems. The most famous is Fourier's trigonometric system, with basis functions being the trigonometric functions $\sin nx$ and $\cos nx$ for varying values of $n = 1, 2, \ldots$. In this setting we define the inner product of two functions f and g by

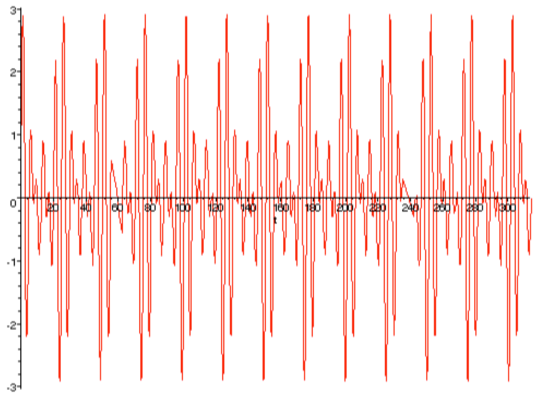
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$$

and for example we have that

$$\langle \sin nx, \cos mx\rangle = 0$$

where m and n are integers. Also, similar results apply regardless of which of the given trigonometric functions we choose, as long as they didn't coincide.

An important application of trigonometric functions is to interpret them as sound waves of varying frequencies.



The figure shows an illustration of a three-tone major chord. This wave function can be "tested" against pure sinusoidal waves of varying frequencies. By testing we mean that we compute the inner product of this wave function with the basis functions $\sin nx$. The basis functions are mutual orthogonal, thus the inner product will vanish except when we precisely "hit" one of the sinusoidal functions that constitute the three-tone chord. This will give a (theoretical) procedure for decomposing the sound to its basic ingredients. Some gifted people are able to do this in practice, cf. the anecdote about the 14-year-old Mozart who wrote down the score of Allegri's Miserere after listening to it once during the Wednesday service in the Vatican.

What we really wish for in an orthogonal function system is **completeness**. Suppose we have given an orthogonal function system $\phi_1, \phi_2, \phi_3, \ldots$ (which we can assume is orthonormal by just dividing out each function by the inner product of the function by itself). We would like to decompose an arbitrary function f relative to this system. We compute the coefficients of the components

$$\langle f, \phi_j \rangle$$

and put them together to give a new function

$$\overline{f} = \sum_j \langle f, \phi_j \rangle \, \phi_j$$

Because of the orthonormality of the system this function will have the nice property

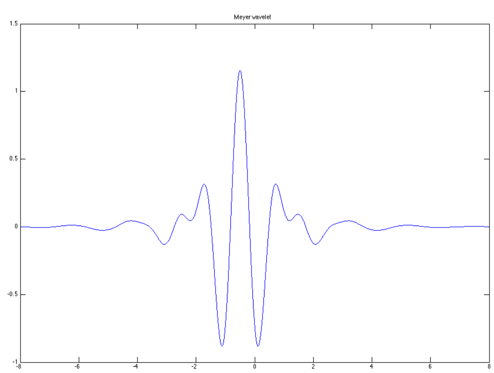
$$\overline{\overline{f}} = \overline{f}$$

The question is whether we have $\overline{f} = f$. If this is (almost) true, we say that the function system is complete. This is not always the case, but if it is true we are able to deduce important results about the system.

In this case the basis functions are interpreted as stationary waves of varying frecquencies. The system is thus well-behaved for decomposing functions of similiar form as the basis functions. For other functions, not so regular and with sudden changes, the trigonometric basis functions are not equally suited for the decomposition. Suppose that the function in some way or another describes the digitalization of a photography. Then the parts of the picture which are rather homogenous, e.g. the sky, can be considered to be stationary. But contours and contrasts will behave very non-stationary.

A tool to handle such rapid changes is to modify Fourier's trigonometric basis functions with a certain window technique, i.e. we only consider a small part of a harmonic oscillation. To be able to detect sudden changes in a signal, or in the describing function, we need a narrow window. The narrower the window is, the more precisely we can locate the change. A consequence is that what we "see" through the window might not be enough to decide the frequency. If we reduce the window to only one point, we can of course decide exactly the time of the change, but we can not determine the frequency, since one value is not enough to determine a function. To obtain a good result we need to vary the window width and not only the frequencies, which makes the procedure rather complicated and inefficient.

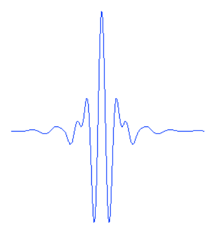
The modern wavelet theory, introduced by Jean Morlet and developed into a robust mathematical theory by Yves Meyer, handles these problems better.



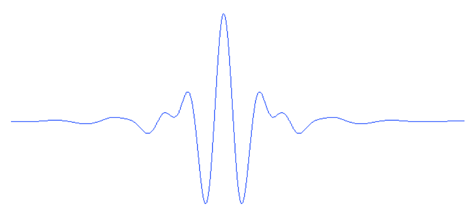
Meyer's wavelet

The basis for the theory is a "mother wavelet". This function is used to create the orthogonal function system. Thus the basis functions all have the same shape. The mother wavelet may be a scaled part of some wave equation, as in Meyer's wavelet. In fact wavelet means "small wave". If we want to increase the width of the wavelet, we just stretch the graph. Keeping the shape means that increasing the width will lower the frequency and vice versa.

The window-scaled trigonometric basis functions are well suited for handling nonstationary signals with sudden changes. And fortunately they form an orthogonal function system.



Meyer's wavelet, reduced width



Meyer's wavelet, extended width

When Meyer had established his general theory, the ground was prepared for constructing new wavelets, perfectly fitted for certain applications. As long as one used the motherwavelet method, one didn't have to worry about the behavior of the associated function system.