THE
ABEL
PRIZE
2015

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2015 to **John F. Nash, Jr.**, Princeton University and **Louis Nirenberg**, Courant Institute, New York University

“for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis.”

Partial differential equations are used to describe the basic laws of phenomena in physics, chemistry, biology, and other sciences. They are also useful in the analysis of geometric objects, as demonstrated by numerous successes in the past decades.

John Nash and Louis Nirenberg have played a leading role in the development of this theory, by the solution of fundamental problems and the introduction of deep ideas. Their breakthroughs have developed into versatile and robust techniques, which have become essential tools for the study of nonlinear partial differential equations. Their impact can be felt in all branches of the theory, from fundamental existence results to the qualitative study of solutions, both in smooth and non-smooth settings. Their results are also of interest for the numerical analysis of partial differential equations.

Isometric embedding theorems, showing the possibility of realizing an intrinsic geometry as a submanifold of Euclidean space, have motivated some of these developments. Nash’s embedding theorems stand among the most original results in geometric analysis of the twentieth century. By proving that any Riemannian geometry can be smoothly realized as a submanifold of Euclidean space, Nash’s smooth (C^∞) theorem establishes the equivalence of Riemann’s intrinsic point of view with the older extrinsic approach. Nash’s non-smooth (C^1) embedding theorem, improved by Kuiper, shows the possibility of realizing embeddings that at first seem to be forbidden by geometric invariants such as Gauss curvature; this theorem is at the core of Gromov’s whole theory of convex integration, and has also inspired recent spectacular advances in the understanding of the regularity of incompressible fluid flow. Nirenberg, with his fundamental embedding theorems for the sphere S^2 in R^3, having prescribed Gauss curvature or Riemannian metric, solved the classical problems of Minkowski and Weyl (the latter being also treated, simultaneously, by Pogorelov). These solutions were important, both because the problems were representative of a developing area, and because the methods created were the right ones for further applications.

Nash’s work on realizing manifolds as real algebraic varieties and the Newlander-Nirenberg theorem on complex structures further illustrate the influence of both laureates in geometry.

Regularity issues are a daily concern in the study of partial differential equations, sometimes for the sake of rigorous proofs and sometimes for the precious qualitative insights that they provide about the solutions. It was a breakthrough in the field when Nash proved, in parallel with De Giorgi, the first Hölder estimates for solutions of linear elliptic equations in general dimensions without any regularity assumption on the coefficients; among other consequences, this provided a solution to Hilbert’s 19th problem about the analyticity of minimizers of analytic elliptic integral functionals. A few years after

Nash's proof, Nirenberg, together with Agmon and Douglis, established several innovative regularity estimates for solutions of linear elliptic equations with L^p data, which extend the classical Schauder theory and are extremely useful in applications where such integrability conditions on the data are available. These works founded the modern theory of regularity, which has since grown immensely, with applications in analysis, geometry and probability, even in very rough, non-smooth situations.

Symmetry properties also provide essential information about solutions of nonlinear differential equations, both for their qualitative study and for the simplification of numerical computations. One of the most spectacular results in this area was achieved by Nirenberg in collaboration with Gidas and Ni: they showed that each positive solution to a large class of nonlinear elliptic equations will exhibit the same symmetries as those that are present in the equation itself.

Far from being confined to the solutions of the problems for which they were devised, the results proved by Nash and Nirenberg have become very useful tools and have found tremendous applications in further contexts. Among the most popular of these tools are the interpolation inequalities due to Nirenberg, including the Gagliardo-Nirenberg inequalities and the John-Nirenberg inequality. The latter governs how far a function of bounded mean oscillation may deviate from its average, and expresses the unexpected duality of the BMO space with the Hardy space H^1. The Nash-De Giorgi-Moser regularity theory and the Nash inequality (first proven by Stein) have become key tools in the study of probabilistic semigroups in all kinds of settings, from Euclidean spaces to smooth manifolds and metric spaces. The Nash-Moser inverse function theorem is a powerful method for solving perturbative nonlinear partial differential equations of all kinds. Though the widespread impact of both Nash and Nirenberg on the modern toolbox of nonlinear partial differential equations cannot be fully covered here, the Kohn-Nirenberg theory of pseudo-differential operators must also be mentioned.

Besides being towering figures, as individuals, in the analysis of partial differential equations, Nash and Nirenberg influenced each other through their contributions and interactions. The consequences of their fruitful dialogue, which they initiated in the 1950s at the Courant Institute of Mathematical Sciences, are felt more strongly today than ever before.