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HISTORICAL SURVEY

NIELS HENRIK ABEL AND THE BIRTH OF FRACTIONAL CALCULUS

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Dedicated to Professor Virginia Kiryakova on the occasion of her 65th birthday and the 20th anniversary of FCAA

Abstract

In his first paper on the generalization of the tautochrone problem, that was published in 1823, Niels Henrik Abel presented a complete framework for fractional-order calculus, and used the clear and appropriate notation for fractional-order integration and differentiation.

MSC 2010: Primary 26A33; Secondary: 01A55

Key Words and Phrases: fractional calculus, tautochrone, Caputo derivative, time scales, history of mathematics

1. **Introduction**

Niels Henrik Abel's life was too short. He was born on August 5, 1802 (that is, exactly 215 years ago) and he passed away on April 6, 1829, at the age of less than 27 years. Nowadays, many graduate students of a similar age are still pursuing their degrees. Nevertheless, during his short career he had made several monumental mathematical contributions.

His most famous result is, of course, the proof that a general algebraic equations of the fifth degree cannot be solved in radicals.

In addition, in every textbook on calculus we can find the Abel's tests for the convergence of an infinite series (for number series, and for power series).

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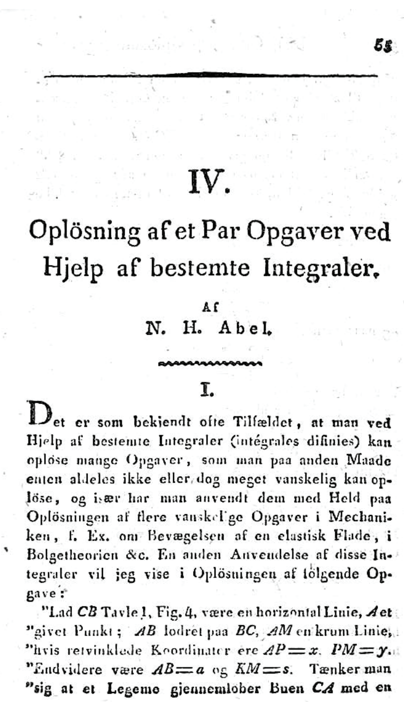
Another of his contributions was the doubly-periodic functions that are now called the Abel elliptic functions.

His name is also associated with several smart transformations, such as Abel's summation by parts, as well as with Abel's inequality, Abel's binomial expansion, the Abelian means for divergent series, and many other elegant results and formulas, without which one cannot imagine many fields of the today's mathematics.

However, one of Abel's most fascinating inventions, the formulas for fractional-order integration and differentiation, has remained practically unnoticed. We found these formulas working on a totally different topic, where we needed to take a closer look at the tautochrone problem. In reading Abel's papers on this topic we discovered that in solving the generalization of the tautochrone problem, Niels Henrik Abel had also developed a complete framework of what is now called the fractional calculus, or differentiation and integration of arbitrary real order.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IV.

Oplösning af et Par Opgaver ved Hjelp af bestemte Integraler.

Af

N. H. Abel.

I.

Det er som bekiendt ofte Tilfældet, at man ved Hjelp af bestemte Integraler (intégrales difinies) kan oplöse mange Opgaver, som man paa anden Maade enten aldeles ikke eller dog meget vanskelig kan oplöse, og især har man anvendt dem med Held paa Oplösningen af flere vanskelige Opgaver i Mechaniken, f. Ex. om Bevægelsen af en elastisk Flade, i Bolgetheorien &c. En anden Anvendelse af disse Integraler vil jeg vise i Oplösningen af fölgende Opgave:

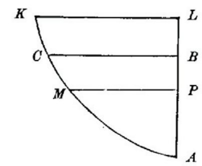
"Lad CB Tavle 1, Fig. 4, være en horizontal Linie, A et "givet Punkt; AB lodret paa BC, AM en krum Linie, "hvis retvinklede Koordinater ere $AP=x$. $PM=y$. "Endvidere være $AB=a$ og $KM=s$. Tænker man "sig at et Legeme gjennemlober Buen CA med en

In his first article ([1], 1823), the first page of which is shown here, N. H. Abel introduced fractional-order integration in the form that is currently known as the Riemann-Liouville fractional integral, and fractional-order differentiation in the form that is currently known as the Caputo fractional derivative. Abel's solution of the considered problem is, in fact, the proof that these two operators are mutually inverse. This means that N. H. Abel, who was only 21 years of age at the time of the publication of his paper, was the father of the complete fractional-order calculus framework.

For reasons that remained unknown, in his second paper ([2], 1826), Abel abandoned this line of theoretical development and its applications, and never returned to it. In the following we take a look at Abel's reasoning in more detail.

2. **Abel's 1823 paper**

First of all, it should be noted that Abel considered a more general problem than the tautochrone (in the quotation below, which is translated and transcribed from [1], ψa means in today's notation $\psi(a)$):



> "Suppose that CB is a horizontal line, A is a setpoint, AB is perpendicular to BC, AM ia a curve with rectangular coordinates $AP = x$, $PM = y$. Moreover $AB = a$, $AM = s$. It is known that as a body moves along an arc CA, when the initial velocity is zero, that the time T, which is necessary for the passage, depends on the shape of the curve, and on a. One has to find the definition of a curve KCA, for which the time T is a given function of a, for example ψa."

The picture on the right appeared in the French translation in [5]. For this problem Abel obtains the following equation:

$$\psi a = \int \frac{ds}{\sqrt{a-x}} \quad (\text{from } x = 0, \text{ to } x = a),$$

and then continues (we added the equation numbers for convenience):

> "Instead of solving this equation, I will show how one can derive s from the more general equation
>
> $$\psi a = \int \frac{ds}{(a-x)^n} \quad (\text{from } x = 0, \text{ to } x = a) \tag{1}$$
>
> where n has to be less than 1 to prevent the infinite integral between two limits; ψa is an arbitrary function that is not infinite, when a equals to zero."

Abel looks for the unknown function $s(x)$ in the form of a power series, and after term-by-term operations with that series uses the properties of the gamma function (paying credits to A.-M. Legendre [3], who studied and summarized the properties of the Euler's gamma function, introduced the notation $\Gamma(z)$ and gave the function its name). After several pages of manipulations with the power series he arrives at the solution of equation (1):

$$s = \frac{\sin n\pi}{\pi} x^n \int \frac{\psi(xt)dt}{(1-t)^{1-n}} \quad (t = 0, \ t = 1), \tag{2}$$

and he calls this "mærkværdige Theorem" – a remarkable theorem.

Indeed, it is remarkable, since Abel further examines the obtained formulas from other viewpoints and mentions that "... s can be expressed in the different way":

$$s = \frac{1}{\Gamma(1-n)} \int^n \psi x.dx^n = \frac{1}{\Gamma(1-n)} \frac{d^{-n}\psi x}{dx^{-n}}\,. \tag{3}$$

In spite of the archaic historical notation, we easily recognize that here Abel uses two expressions for the fractional-order integral: one is the derivative of negative order, and the other is the symbol that later appears in Liouville's works, namely $\int^n \psi x.dx^n$. This means that Abel understood that he unified the notions of integration and differentiation, and that he extended them to non-integer orders !

Writing the equation (1) as

$$\psi(t) = \int\limits_0^t \frac{s'(x)dx}{(t-x)^n}\,, \tag{4}$$

we observe that Abel's equation is nothing else but (up to a constant coefficient) the nowadays famous and widely used Caputo fractional derivative of real order n $(0 \le n < 1)$, and the solution of Abel's equation is simply the inverse operation – fractional-order integral of the same order n.

In his 1823 paper [1] Abel freely uses non-integer orders even in the text, for example: "Differentieres Værdien for s nGange, saa faaer man ..." is literally translated as "En différentiant n fois de suite la valeur de s, on obtient ..." [5], or "Differentiating n times the value of s, one obtains ...", and n here is non-integer ! Moreover, returning to the consideration of the original equation, Abel writes:

> "If $n = \frac{1}{2}$, one obtains
>
> $$\psi a = \int \frac{ds}{\sqrt{a-x}}, \quad (x=0,\, x=a),$$
>
> and
>
> $$s = \frac{1}{\sqrt{\pi}} \frac{d^{-\frac{1}{2}}\psi x}{dx^{-\frac{1}{2}}} = \frac{1}{\sqrt{\pi}} \int^{\frac{1}{2}} \psi x.dx^{\frac{1}{2}}.$$
>
> This is the equation of the desired curve, when the time is ψa."

And at this moment Abel is ready to simply invert the fractional integral by applying fractional differentiation to both sides of the equation and by using the understanding (and the fact), that fractional-order integration and differentiation are mutually inverse operations:

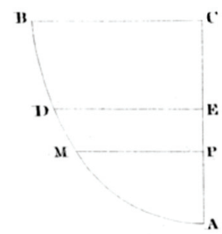
"One obtains from this equation

$$\psi x = \sqrt{\pi}\frac{d^{\frac{1}{2}} s}{dx^{\frac{1}{2}}}\,,$$

therefore: if the curve equation is $s = \psi x$, then the time that the body uses to pass the arc, the height of which is a, equals $\sqrt{\pi}\dfrac{d^{\frac{1}{2}}\psi a}{da^{\frac{1}{2}}}$."

3. **Abel's 1826 paper**

We speculate that Abel was not very satisfied with the manipulations of the power series in his 1823 paper [1]; maybe he considered them too lengthy, or not well justified, or maybe simply not so elegant. In either case, in his paper dated 1826 [2], that is three years after his first paper on the tautochrone problem, he presented a different method of solution, which was based on using the properties of the Euler gamma function. The figure on the right is from the original paper.



First, he derives the following formula (written here in Abel's notation):

$$f\,x = \frac{\sin n\pi}{\pi}\int_0^x \frac{da}{(x-a)^{1-a}}\int_0^a \frac{f'z\,dz}{(a-z)^n}\,, \tag{5}$$

and today we easily recognize here the fractional-order integral of the Caputo fractional-order derivative. Then Abel uses this formula for obtaining the solution of equation (1):

"Multiplying this equation [*that is, (1)*] by $\dfrac{\sin n\pi}{\pi}\cdot\dfrac{da}{(x-a)^{1-n}}$, and integrating it from $a = 0$ to $a = x$, one obtains

$$\frac{\sin n\pi}{\pi}\int_0^x \frac{\varphi a.da}{(x-a)^{1-n}} = \frac{\sin n\pi}{\pi}\int_0^x \frac{da}{(x-a)^{1-n}}\int_0^a \frac{ds}{(a-x)^n}\,.$$

Therefore, accordingly to the equation (1) [*here (5)*]

$$s = \frac{\sin n\pi}{\pi}\int_0^x \frac{\varphi a\,da}{(x-a)^{1-n}}\,."$$

In this paper Abel also considers the case of $n = \frac{1}{2}$, as he did three years earlier:

"Considering for now $n = \frac{1}{2}$, one obtains

$$\varphi\,a = \int_0^a \frac{ds}{\sqrt{(a-x)}}$$

and

$$s = \frac{1}{\pi} \int_0^x \frac{\varphi a \, da}{\sqrt{(x-a)}} \, .\text{”}$$

The 1826 paper [2] is shorter, and the derivation of the solution of equation (1) is concise and looks more elegant; but the fractional-order calculus, which was, in fact, developed in Abel's 1823 paper, disappeared ... Abel shortened and improved the solution of a particular problem, but lost the whole theory of the fractional-order calculus.

4. **Space and time scales**

The tautochrone problem, considered by Abel, also has another aspect, that remained unnoticed. In fact, it provides an example of a whole family of dynamical processes, which, so to say, 'live' in the same time scale. Indeed, the time necessary to reach the destination from each point of the arc of particular shape is the same. However, the spatial scale is different for each starting point, as the distance passed from the start to the end is different for each point on the arc.

5. **Conclusion**

It is not clear why Niels Henrik Abel abandoned the direction of research so nicely formed in his 1823 paper [1], and one can only guess the reasons. Abel had all the elements of the fractional-order calculus there: the idea of fractional-order integration and differentiation, the mutually inverse relationship between them, the understanding that fractional-order differentiation and integration can be considered as the same generalized operation, and even the unified notation for differentiation and integration of arbitrary real order.

It is easy to propose the following hypothesis regarding why the mathematical community did not pay proper attention to Abel's invention of the fractional-order calculus. The original paper was published in the Danish language in the Norwegian journal *Magazin for Naturvidenskaberne*, so the readership was very limited. Thus, Abel's 1823 paper was not included in his posthumous "Œuvres complètes de N. H. Abel" compiled by B. Holmboe and published in Christiania in 1839 [4], while a translation of Abel's 1826 paper [2] was included there as Chapter IV. Later, in 1881, L. Sylow and S. Lie compiled another "Œuvres complètes de Niels Henrik Abel" [5], also printed in Christiania; they included a translation of Abel's 1823 paper as Chapter II, and a translation of Abel's 1826 paper as Chapter IX. However, apparently the initial picture of Abel's scientific work created by Holmboe's collection [4] lasted too long, and Abel's 1823 paper remained in the shadow of his elegant 1826 paper that became known to the mathematical community earlier, as it was published in the already famous Crelle's journal.

In Abel's 1823 paper [1] we find the notation that was later used by Liouville for fractional-order integration, and the definition of fractional differentiation that is now called the Caputo fractional differentiation. It took 144 years until Professor Michele Caputo came up with his definition in his paper in 1967 [6] (we need to mark nowadays its 50 years) and in his book [7] in 1969. However, as Hans Selye wrote, the discovery *"is not to see something first, but to establish solid connections between the previously known and the hitherto unknown"* [8]. Abel's discovery of the fractional calculus remained unnoticed in his days (even by himself), so Professor Caputo's name is therefore properly associated with this definition, as in his works he demonstrated numerous applications of fractional-order differentiation to viscoelasticity, geophysics, and other fields.

Acknowledgments

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