John F. Nash, Jr. and Louis Nirenberg, Abel Prize Laureates 2015

INTRINSIC OR EXTRINSIC PROPERTY?

In mathematics, the distinction between intrinsic and extrinsic properties of a geometric object is very important. An intrinsic property will not change, regardless of how we represent the object, an extrinsic property does not have the same immutability.

A dictionary gives this definition of the word intrinsic: *Belonging to a thing by its very nature*. Thus an intrinsic property is a property of an object which is independent of how or in which context the object is represented. The opposite is an extrinsic property. The length of a curve is an intrinsic property, while the curvature is extrinsic. Just think of the curve as an inelastic rope. The length is constant, but the curvature depends on how we coil the rope.

The curvature of a curve was introduced as early as in the fourteenth century, and more thoroughly understood by Leonhard Euler several hundred years later. But the deeper understanding of the nature of such geometric quantities was manifested only when Carl Friedrich Gauss started looking at surfaces and raised the question of intrinsic vs. extrinsic properties. It is not surprising that Gauss discussed curvature in this case; unlike the situation for curves, curvature is an intrinsic property of a surface.

In the *Disquisitiones generales circa superficies curvas* from 1827, Gauss formulated the so-called Theorema Egregium:

> *Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet.*

("If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.") In modern words this is formulated: *Gaussian curvature is an intrinsic property of a surface.* The theorem is *remarkable* since curvature originally was defined extrinsically.

Carl Friedrich Gauss (1777-1855), painted by Christian Albrecht Jensen

Influenced by Gauss' geometry on a surface in Euclidean 3-space, Bernhard Riemann (1826-1866) introduced in 1854 Riemannian geometry. Riemannian geometry generalizes Euclidean geometry, and it immediately became a very important tool for the development of geometric and physical ideas in the twentieth century. A Riemannian manifold is a geometric object which is defined intrinsically, i.e. without reference to any embedding of the object in an ambient space.

Since the celebrated embedding theorem of John F. Nash Jr. allows geometers to view each Riemannian manifold as a submanifold of a Euclidean space, the problem of discovering simple sharp relationships between intrinsic and extrinsic invariants of a submanifold is one of the most fundamental problems in submanifold theory.

In the beginning every property was thought of as being extrinsic. Shortly after Gauss drew our attention to the difference between intrinsic and extrinsic properties, Riemann followed the track and redefined geometric objects in an intrinsic way. The consequence was that geometric objects could not necessarily be globally realized in any ambient space. The embedding theorems of the 1950's suggested a link back in time, to realize a Riemannian manifold as a submanifold of Euclidean space, and open up for giving extrinsic definitions of intrinsic properties.