John F. Nash, Jr. and Louis Nirenberg, Abel Prize Laureates 2015

THE NASH-KUIPER THEOREM

During the mid-50's John F. Nash, Jr. published two papers, describing two theorems known as the Nash embedding theorems. Both papers deal with so-called isometric embeddings of geometrical objects into Euclidean space, i.e. embeddings that preserve the length of every path in the geometrical object.

The two theorems are very different from each other; the first one, referred to as the C^1 theorem, has a very simple proof and leads to some very counterintuitive conclusions, while the proof of the second one, the C^k theorem, is very technical, but the result is not quite surprising. The C^1 theorem was published in 1954, and was extended by Nicolaas Kuiper the next year. The C^k theorem was published in 1956.

The Nash-Kuiper Theorem. Let (M, g) be a Riemannian manifold of dimension m and $f : M \to \mathbb{R}^n$ a short C^∞-embedding into Euclidean space $\mathbb{R}^n$, where $n \geq m+1$. Then for arbitrary $\epsilon > 0$ there is an embedding $f_\epsilon : M \to \mathbb{R}^n$ in class C^1, which is

(i) isometric: for any two tangent vectors $v, w \in T_x(M)$,

$$g(v, w) = \langle df_\epsilon(v), df_\epsilon(w) \rangle$$

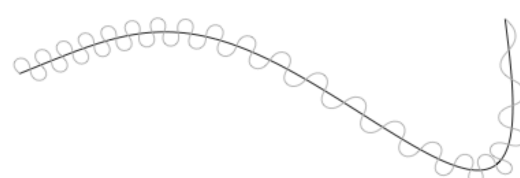
(ii) ϵ-close to f:

$$|f(x) - f_\epsilon(x)| < \epsilon, \;\; \forall x \in M.$$

In the proof of the Nash-Kuiper theorem a short embedding of a Riemannian manifold into a Euclidean space is converted into a C^1-isometric embedding. A short embedding is a map that shortens the length of curves. The following sketch of the proof of the Nash-Kuiper theorem is a simplified version, conducted in dimension one. The geometrical object is a circle and the target manifold is the Euclidean 2-plane.

Let C be a plane regular curve, parametrized by a map $\mathbf{r} : [0,1] \to \mathbb{R}^2$. The curve is traversed at a speed $v_0(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$. Let $v(t)$ be another speed function, exeeding $v_0(t)$, i.e. $v(t) \geq v_0(t)$ for all $t \in [0,1]$. The topic of the one-dimensional Isometric Problem is whether it is possible to construct a new regular curve C', parametrized by $\mathbf{r}' : [0,1] \to \mathbb{R}^2$ such that the curve is traversed at a speed v. This is of course no problem, but we have imposed an additional requirement, the new curve should be as close to the original curve as we want, to be precise $\|\mathbf{r}'(t) - \mathbf{r}(t)\| \leq \epsilon$ for any choice of $\epsilon > 0$.

The answer to this question is yes, it is possible, due to the Nash-Kuiper Theorem.

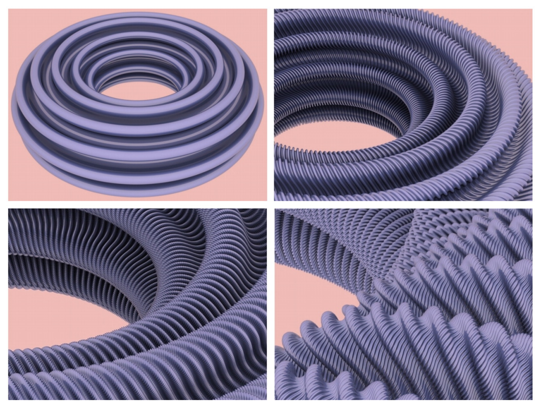


The new curve C' is build on the original by adding an oscilating curve in the normal direction of the curve. If the added curve has sufficiently short wavelength and sufficiently small amplitude, the two requirements, speed and proximity, are fullfilled. The shorter the wavelength, the longer the curve. To be sure that the distance to the original curve is sufficiently small, the amplitude must be small. But it is possible to establish a curve, sufficiently long so that it will be traversed at the given speed.

(*Photo by Bernard Thompson*)

An illustrating example is a cyclist cycling up a steep uphill. The original curve is tracked out by the rear-wheel, the new curve by the front-wheel. With many rapid oscillations, the front-wheel will track out a much longer distance than the rear-wheel in the same time period. Thus the front-wheel curve is traversed at a higher speed than the rear-wheel curve.

Whereas the one-dimensional version of Nash's theorem is rather intuitive, the two-dimensional version is more or less counter-intuitive, as the following illustration shows. Start with a piece of paper and turn it into a cylindrical shape. This is easy. The next step is the hard part. To turn the cylinder into a donut shaped surface without stretching or tearing the paper. Intuitively this seems to be impossible. The outer circumference of the donut is much longer than the inner, but in the original cylinder they are of the same length. By Nash's theorem this is never the less possible, at least theoretically. Nash proved the theorem in 1954, but it was only in 2012 a multidisciplinary team in France, the HEVEA project, was able to image the process where the cylinder is bent into a donut, in an isometric way. The above images illustrate the process; the paper is warped by an infinite sequence of waves, piling up to a donut surface in such a way that the originally piece of paper is kept intact.



Images of an isometric embedding of a flat torus in $\mathbb{R}^3$. *(Source: HEVEA Project/PNAS)*