John F. Nash, Jr. and Louis Nirenberg, Abel Prize Laureates 2015

NEVER CHANGE A GIVEN DISTANCE ...

The Abel Commitee explains the choice of this year's Abel Prize Laureates: "Nash's embedding theorems stand among the most original results in geometric analysis of the twentieth century." . . . "Nirenberg, with his fundamental embedding theorems for the sphere S^2 in $\mathbb{R}^3$, having prescribed Gauss curvature or Riemannian metric, solved the classical problems of Minkowski and Weyl."

Neurons are not evenly distributed in the human body. Some parts of the body, like our hands, our face and our tongue are much more sensitive to sensations than other parts. The body has the highest density of neurons in those parts. A function that measures the density of neurons is an example of what mathematicians call a metric. Another example of a metric is the so-called Euclidean metric, named after the ancient Greek mathematician Euclid. The Euclidean metric measures ordinary distances between points and area of any region of a surface. In a paper from 1916 Hermann Weyl asked the following question: *Is it always possible to realise an abstract metric on the 2-sphere of positive curvature by an isometric embedding in $\mathbb{R}^3$?* If you think of the neuron density metric as Weyl's abstract metric and the human body as the 2-sphere, then the weird body in figure 1 illustrates the positive answer to Weyl's question. The different sizes of the various body parts correspond to the neuron density.

Figure 1: The different size of the parts of the body reflects the density of neurons. (Source: Natural History Museum, London)

Long time before spacecrafts provided us with images of the earth, our forefathers concluded that our planet is round. They based this knowledge on observations done on the surface of the earth. By performing smart observations and correct measurements, they were able to conclude that the earth could not be flat. If you fix a point on a flat surface and you walk a circular path at a given distance R, the path should be $2\pi R$ long. But if you measure carefully on the earth's surface you will find that the perimeter is a bit shorter. A theoretical computation then tells you that the earth's surface has positive curvature, i.e. locally it looks like a sphere.

The fact that it is possible to say anything about the curvature, merely by observations performed on the surface, was formulated by the great mathematician Carl Friedrich Gauss in 1827, in what is called the Gauss' Theorema Egregium, *the remarkable theorem.* The theorem says that the Gaussian curvature of a surface can be determined entirely by measuring distances and angles on the surface itself, without further reference to how the surface is embedded in the 3-dimensional space. Curvature is an intrinsic property of the surface, i.e. a property that belongs to the surface by its very nature. Consequently it has to be preserved by any isometric embedding.

In the first embedding theorem of John F. Nash, Jr., published in 1954, he proves that any Riemannian manifold can be isometrically embedded in Euclidean space by a C^1-map. The striking point of a curve-version of this theorem is that any curve in the plane can be arbitrarily prolonged in a smooth way, without self-crossing and as close to the original curve as we want. The prolonged curve looks like the path of the front-wheel of a bicycle climbing a steep hill, when the rear-wheel tracks out the original curve. By increasing the frequency of twists the cyclist can increase the difference between the length of the front- and rear-wheel paths. Unlike the surface case, curvature of a curve does not have to be preserved by an isometric embedding.

Whereas the one-dimensional version of Nash's theorem is rather intuitive, the two-dimensional version is more or less counter-intuitive, as the following illustration shows. Start with a piece of paper and turn it into a cylindrical shape. This is easy. The next step is the hard part. To turn the cylinder into a donut shaped surface without stretching or tearing the paper. Intuitively this seems to be impossible. The outer circumference of the donut is much longer than the inner, but in the original cylinder they are of the same length. By Nash's theorem this is nevertheless possible, at least theoretically. Nash proved the theorem in 1954, but it was only in 2012 a multidisciplinary team in France, the HEVEA project, was able to image the process where the cylinder is bent into a donut, in an isometric way. The images in figure 2 illustrate the process; the paper is wrapped by an infinite sequence of waves, piling up to a donut surface in such a way that the original piece of paper is kept intact.

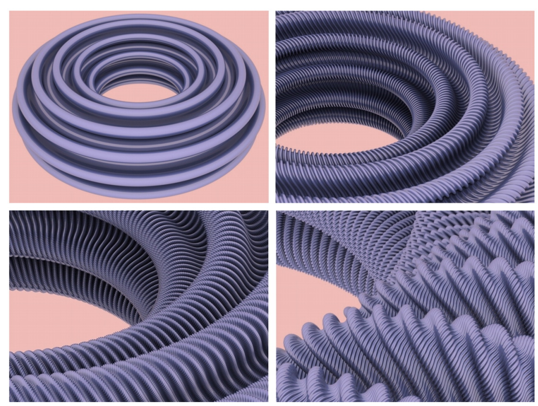

Figure 2: Images of an isometric embedding of a flat torus in $\mathbb{R}^3$. (Source: HEVEA Project/PNAS)