John F. Nash, Jr. and Louis Nirenberg, Abel Prize Laureates 2015

PARTIAL DIFFERENTIAL EQUATIONS - A UNIVERSAL MATHEMATICAL TOOL

Partial differential equations (or PDEs for short) are used to describe the basic laws of phenomena in physics, chemistry, biology, and other sciences. They are also useful in the analysis of geometric objects, as demonstrated by numerous successes in the past decades. John F. Nash Jr. and Louis Nirenberg have played a leading role in the development of this theory, by the solution of fundamental problems and the introduction of deep ideas.

Jean-Baptiste Joseph Fourier (1768-1830) was an orphan, a revolutionary, an advicer for Napoleon Bonaparte and ended up as Permanent Secretary of the French Academy of Sciences. But he is mainly known as a highly influential mathematician and physicist.

In 1822 Fourier published his work on heat flow in *Théorie analytique de la chaleur* (The Analytic Theory of Heat) where he introduced the heat equation,

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

The function $u = u(t, \mathbf{x})$ measures the temperature at a given point at a given time and the equation gives a mathematical model for heat transfer. The equation is a mathematical expression for the fact that in a point where it is colder than the average of the nearby surrondings the temperature will increase over time. This type of differential equation is called a parabolic PDE.

The first bowed string instruments may have originated in the equestrian cultures of Central Asia, but the instrument we today recognize as a violin was developed in Southern Europe during the renaissance through the noble art of violin making of the Stradivari family and others. The virtuosos enthralled their audience and the scientists wondered how the beautiful sound was created by the vibrating string. In 1746, Jean le Rond d'Alembert discovered the one-dimensional wave equation, and within ten years Leonhard Euler had described the phenonema in three dimensions. The wave equation translates the dynamics of a wave into mathematics. It is usually written as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

where $u = u(t, \mathbf{x})$ describes the amplitude of the wave function, as it propagates in time and space. Considering a vibrating string or wave in general, we notice that some times the wave look like $\frown$ and some times it looks like $\smile$. In the first case the wave has started to fall, and in the other case it grows. The mathematical model for this is a hyperbolic PDE.

The parabolic and the hyperbolic PDEs are typical models for dynamical systems, i.e. systems that evolve over time. The distinction between the two types is the speed of the propagation, there is much more vigor in a system where the speed rather than the acceleration is caused by a convexity in the solution. Heat transfer, modelled by a parabolic equation, propagates by infinte speed. Every part of the silver spoon is instantly affected by the hot tea, even if it takes some time before your fingers hurt. Waves propagate at finite speed; the sea is completely calm before the wave enters.

John F. Nash, Jr. and Luis Nirenberg are mainly associated to a third type of PDEs, called elliptic PDEs. The elliptic equations have much in common with hyperbolic equations, but they differ on one crucial point, namely whether there exist a time-like variable. In an elliptic PDE there is no such variable, all coordinates are space-like.

The elliptic partial differential equation will typically look like

$$\nabla^2 u = f$$

with no time-coordinate involved. The solutions of elliptic equations are purely spatial, there is no natural way to introduce a time coordinate. Elliptic equations therefore typically model static physical problems.

The Abel Prize Commitee emphasizes the influence of the Laureates in the development of the theory of elliptic PDEs: *Regularity issues are a daily concern in the study of partial differential equations, sometimes for the sake of rigorous proofs, and sometimes for the precious qualitative insights that they provide about the solutions. It was a breakthrough in the field when Nash proved, in parallel with De Giorgi, the first Hölder estimates for solutions of linear elliptic equations in general dimensions without any regularity assumption on the coefficients; among other consequences, this provided a solution to Hilbert's 19th problem about the analyticity of minimizers of analytic elliptic integral functionals. A few years after Nash's proof, Nirenberg established, together with Agmon and Douglis, several innovative regularity estimates for solutions of linear elliptic equations with* L^p *data, which extend the classical Schauder theory and are extremely useful in applications where such integrability conditions on the data are available. These works founded the modern theory of regularity, which has since grown immensely, with applications in analysis, geometry and probability, even in very rough, non-smooth situations.*