Abel Prize 2008

Solution to Rubik's Cube in a few Seconds, using Group Theory

From time to time Rubik's cube has a popularity peak. Many people get more or less addicted and spend a lot of time trying to find a solution. Most of us give up after some frustrating attempts. It seems impossible to collect all the blue fields on one side at the same time as the reds and the yellows are not spread all over.

Having put the cube away, a young student appears on the Nine o'clock News, solving the problem in 20 seconds. "OK", we say, "students of today have bright minds", and we accept the situation without loosing our confidence. We might even accuse the student of spending too much time on this worthless activity instead of studying the curriculum. But the thought crosses our mind: How do they do it? Then a three-years-old Chinese in his baby-chair shows us how to solve the cube with his tiny fingers and convinces us that there must be some sort of system.

The solution is group theory.

Rubik's cube contains a kernel on which the rest of the subcubes can be moved around in layers. The only visual part of the kernel is the six outer fields in the middle of each face of the cube. Each of these outer fields is surrounded by 8 subcubes which simultaneously is rotated around, and with, the central field. The cube contains 3 x 3 x 3 = 27 subcubes, of which 7 belongs to the kernel and 20 can be moved. The 20 movable subcubes are divided in 8 corner cubes and 12 edge cubes. The corners have three visual faces, the edge cubes have two. A symmetry of the cube has to map corners to corners and edge cubes to edge cubes. There are 8! permutations of the corners and 12! permutations of the edge cubes. The corners have three visual faces, increasing the possibilities by a factor 28, similar for the three faces of the corners. It turns out that not all theoretical possibilities are possible to perform, and we have to divide by a factor of 12. Thus there are all together

$$\frac{1}{12}\, 8!\, 3^8 12! 2^{12} = 43\ 252\ 003\ 274\ 489\ 856\ 000$$

legal symmetries of Rubik's cube. These symmetries form a group structure which we call ***Rubik's group***. As to the Classification of Finite Simple Groups, we mention that Rubik´s group is not simple; it is built up of the simple groups A_{12}, A_8, 7 copies of $\mathbf{Z}_3$ and 12 copies of $\mathbf{Z}_2$.

The key to a quick solution of Rubik's cube is knowledge of certain subgroups of Rubik´s group. One single rotation changes 20 out of the 48 movable fields and fixes the other 28. Composing several rotations in a clever manner increases the number of fixed fields. This is precisely the skill of the cube experts. They know, and remember, sequences of rotations that fix a huge number of fields, and they use these sequences to change more and more fields into the right position, without disturbing the ones they have already placed. From a group theoretic point of view this is not so complicated. To remember and to accomplish the sequences is quite another business. Rubik´s cube is not only a nice example of applied group theory, it is definitely an evidence of the fact that theory is one thing, to put it into practice is quite another.