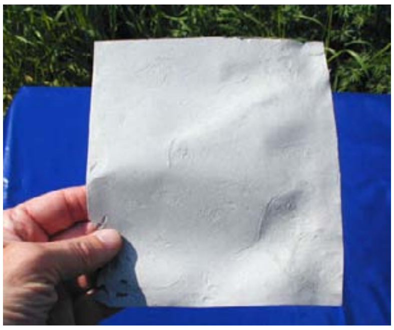
Abel Prize 2009

Differential Geometry

Differential geometry is the study of smooth, curved objects. In this popularized exposition we try to illustrate the subject.

Consider a sheet of paper. It is flat, but possible to roll up, although it has a certain inflexibility. When it lies flat on a desk, it has perfect straight lines along each direction. Now, pick up the sheet, and roll it up. That is, you are allowed to roll the paper however you wish, but you are not allowed to fold or tear it. You should easily be able to make it into a cylinder or a cone. Observe that no matter how you do this, at every point of your sheet of paper there will always be a direction along which perfect straight lines exist. It seems as if you cannot completely destroy the flatness of your sheet of paper. The sheet has **zero curvature.**



Now consider the folowing situation. The president of Exxon has a vision about the gas stations in the US being equally distributed all over the continent. He draws circles around the headquarter, of radius 100 miles, 200 miles, 300 miles and so on. Then he asks his staff to count the number of gas stations within each circle. The area of a circle is proportional to the square of the radius, and the number of stations should grow like 1, 4, 9, 16, ... After some time the result is communicated to the president. It seems that the gas stations are concentrated around the headquarter. But the strange fact is that if they draw the circle around a completely different center, the same phenomena occurs. There are always more stations close to the center of the circle than further out. Actually this is no paradox, it reflects the **positive curvature** of the sphere-shaped surface of the earth.

Now consider your arm. The shape of your shoulder joint allows you a certain rotational degree of freedom. Furthermore, the pair of bones in your forearms, the radius and ulna, gives your wrists the necessary rotational freedom for turning doorknobs. Consider the movements where you are allowed to rotate your shoulder, but not your wrists. Try this. Hold out your arm perfectly straight, in front of you, with your hand opened, fingers together, palm down. Keeping everything rigid, rotate your shoulder so that your fingers are pointing straight up, as if you were to ask a question in class. Rotate rigidly again until your arm is once again in a horizontal position, pointing right out at your side, as if you were half-crucified. Now bring your arm back in front of you. Your palm should now be pointing sideways instead of downwards as it originally was. Whar has happened? You have rotated your wrist by moving your arm along a spherical triangle, but at no point did you actually use the extra rotational freedom afforded by the pair of bones in your forearm. Use it now. Keeping your arm rigid, rotate your wrist until your palm faces down. Feel the motion of muscles that you didn't use before. Because you moved your hand along a triangle lying on the sphere described by the radius of your arm, the curvature of the sphere turned your hand when you brought it back to its original position,

even though you didn't rotate your wrist during these motions but kept it rigid relative to the path of motion. If you had tried the same trick this time moving along a zero curvature plane, your hand would have had the same orientation when you moved it back to its original position in the plane. This is an example of what it is like to **parallel transport** your hand along a spherical triangle.

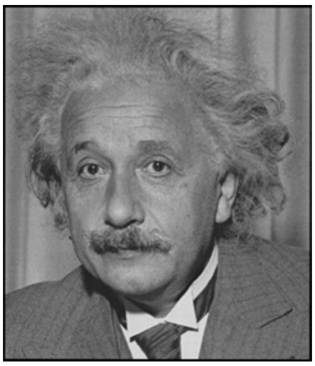
Perhaps all this hard work has made you hungry. Let us consider a doughnut. Smooth and nice with a hole in the middle. Before you eat it, let us examine the curved structure of the surface of the doughnut. Inside, near the hole, we say that the curvature is negative, i.e. the doughnut curves in two different directions, like a saddle. On the outer part, the curvature is said to be positiv, curving the same way in all directions. Adding up we find that the total curvature of our doughnut equals zero, and that this happens with any other sort of pastry that has a hole through it. A similar result holds for the total curvature of any bun or a sphere. It adds up to 4π. The general result, called the **Gauss-Bonnet theorem,** says that the total curvature of any smooth curved surface only depends on the number of holes, subtracting 4π for each hole.

Differential geometry is the branch of geometry that concerns itself with smooth curved objects. Differential geometry studies local properties such as measuring distance and curvature, or global properties such as orientability. A first approximation to understanding what differential geometry is about is to understand what it is not about. Differential geometry is not the same as Euclidian geometry. The latter most often deals with objects that are straight and uncurved, such as lines, planes, and triangles, or at most curved in a very simple fashion, such as circles. Differential geometry prefers to consider Euclidean geometry as a very special kind of geometry; that of zero curvature. Nonzero curvature is where interesting things seem to be happening. Historically, it might be possible to divide differential geometry into classical and modern, with the line of demarcation drawn somewhere through Bernhard Riemann's inaugural lecture given in 1854 in Göttingen. This lecture laid the foundation for modern differential geometry, inspiring geometers for many decades. Classical differential geometry began with the study of curved surfaces in space, such as spheres, cones, cylinders, hyperbolic paraboloids, or ellipsoids. A key notion always present in differential geometry is that of curvature. The first person to illuminate this problem was Leonhard Euler (1707-1783), who is in fact associated with every branch of mathematics that existed in the eighteenth century.

Euler can probably be creditted for much of the early explorations in differential geometry, but his influence is not quite as profound as the reverberations that Carl Friedrich Gauss's (1777 - 1855) seminal paper *Disquisitiones generales circa superficies curvas* (General investigations of curved surfaces) from 1827 propagated through the subject. Gauss's paper gives us an almost modern definition of a curved surface, as well as a definition and precise procedures for computing the curvature of a surface, that now bears his name. He also defines the first and second funda-

mental forms of a surface, and the importance of the first has survived to modern-day differential geometry in the form of a Riemannian metric in Riemannian geometry. Using these concepts, and the intrinsic property of the first fundamental form, only depending on the surface itself, and not on how this surface is placed in the surrounding Euclidean space, he proves the ***theorema egregium*** (remarkable theorem). Gauss presented the theorem in this way:
Theorem (Gauss, 1828) *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged*.
The theorem is ”remarkable” because the definition of Gaussian curvature makes direct use of the position of the surface in space. So it is quite surprising that the final result does not depend on the embedding.

From another angle, Albert Einstein (1870-1955) noticed that he needed a new theory of geometry if he was to generalise his theory of relativity to the case of noninertial frames of reference. Once physicists found applications for the differential geometry that mathematicians had been developing for so long, they started to contribute to the subject and develop their own tradition and schools.



The intervention of the physicists enriched and complicated the subject immensely, with mathematicians sometimes working in parallel with the physicists' traditions, sometimes intersecting, sometimes not, as if trying themselves to imitate the same variations of the parallel postulate that their study of manifolds now afforded them. Non-definite metrics such as the Minkowski metric describing the geometry of spacetime, gained prominence. From a different direction, classical and analytical mechanics and its study of mechanical system lead to the birth of symplectic geometry.

Yet another tributary to this river of dreams came a little earlier in the late 19th century with the Norwegian Sophus Lie (1842-1899) who decided to carry out the ideas of Felix Klein (1849-1925) and his Erlanger Programm and consider continuous, differentiable even, groups that could tell us something about the symmetries of the manifolds under scrutiny, these groups also manifolds in their own right. His Lie groups are an important area of modern research in themselves.

But what is Riemannian Geometry? Euclidean Geometry is the study of flat space. Between every pair of points there is a unique line segment which is the shortest curve between those two points. These line segments can be extended to lines. Lines are infinitely long in both directions and for every pair of points on the line, the segment of the line between them is the shortest curve that can be drawn between them. Furthermore, if you have a line and a point which isn't on the line, there is a second line running through the point, which is parallel to the first line (never hits it). All of these ideas can be described by drawing on a flat piece of paper. From the laws of Euclidean Geometry, we get the famous theorems like Pythagorus' Theorem and all the formulas we learn in trigonometry, like the law of cosines.
Now, suppose instead of having a flat piece of pa-

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per, you have a curved piece of paper, like a sphere. A shortest curve between any pair of points on such a curved surface is called a minimal geodesic. You can find a minimal geodesic between two points by stretching a rubber band between them. The first thing that you will notice is that sometimes there is more than one minimal geodesic between two points. There are many minimal geodesics between the north and south poles of a globe. Surfaces like these are harder to study than flat surfaces but there are still theorems which can be used to estimate the length of the hypotenuse of a triangle, the circumference of a circle and the area inside the circle. These estimates depend on the amount that the surface is curved or bent.

Riemannian Geometers also study higher dimensional spaces. The universe can be described as a three dimensional space. Near the earth, the universe looks roughly like three dimensional Euclidean space. However, near very heavy stars and black holes, the space is curved and bent. There are pairs of points in the universe which have more than one minimal geodesic between them. The Hubble Telescope has discovered points which have more than one minimal geodesic between them and the point where the telescope is located. This is called gravitational lensing. The amount that space is curved can be estimated by using theorems from Riemannian Geometry and measurements taken by astronomers. Physicists believe that the curvature of space is related to the gravitational field of a star according to a partial differential equation called Einstein's Equation. So using the results from the theorems in Riemannian Geometry they can estimate the mass of the star or black hole which causes the gravitational lensing.

Gravitational Lens in Abell 2218 HST • WFP