Abel Prize 2009

Geometry - the playground of Mikhail Gromov

The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2009 to Mikhail Leonidovich Gromov, IHÉS, Bures-sur-Yvette, France, ***for his revolutionary contributions to geometry.*** **The Russian-French mathematician Mikhail L. Gromov is one of the leading mathematicians of our time. He is known for important contributions in many areas of mathematics, especially geometry. Mikhail Gromov has over the past 30 years produced profoundly original, general ideas, which have resulted in new perspectives on geometry and other areas of mathematics**

Riemannian geometry

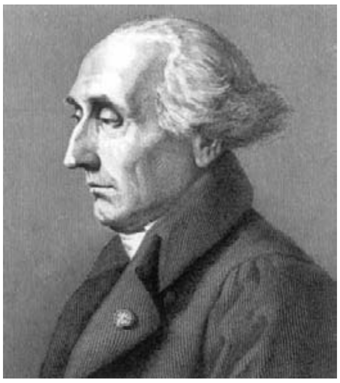
Gromov's work is in the field of differential geometry. Differential geometry is the branch of geometry that concerns itself with smooth curved objects like curves, surfaces or even higher-dimensional manifolds, with various additional structures, for example a Riemannian metric. A Riemannian metric on a surface allows us to measure distances and angles on the surface. A surface equipped with a Riemannian metric is called a Riemannian surface. Notice that the usual Euclidian distance is just one of many possible metrics. Think of the surface as a part of a landscape and introduce a metric that expresses the differences in walking speed in various directions in a given point. In a marsh the values are

low compared to on dry ground. Different values in different directions from a given point reflect variations in the ground in different directions. The distance between two points in this metric is the time taken when walking between the points. One of the fundamental concepts of a Rieman-

nian surface is the Gaussian curvature, first studied by Leonhard Euler (1707-1783) and in greater depth by Carl Friedrich Gauss (1777-1855). A plane with the Euclidian metric has zero Gaussian curvature, while the same plane equipped with the how-difficult-to-walk-metric given above might be rather curvy. Today mathematicians have several different definitions for curvature, all invented to measure how far a surface is from being flat as a plane.

The principal curvatures at a point are the maximum and minimum curvatures of the plane curves obtained by intersecting the surface with planes normal to the tangent plane at the point. The Gaussian curvature at a point is defined to be the product of the principal curvatures at the point; the mean curvature is defined to be their sum.

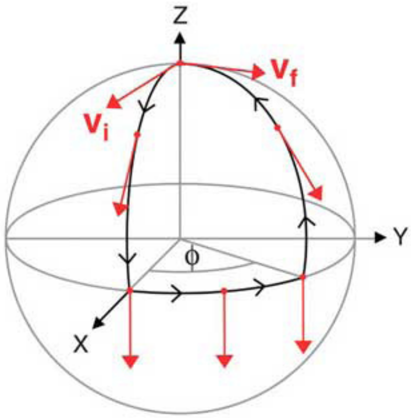
Generally, the most important properties of a surface are those that are defined intrinsically, i.e., determined solely by the distance within the surface as measured along curves on the surface. Surfaces naturally arise as graphs of functions of a pair of variables and before Gauss curvature was defined by a formula involving the defining function. Properties of the surface defined in this way are called extrinsic, the opposite of intrinsic. Gauss showed in his Theorema Egregium ("Remarkable theorem") that in spite of the originally extrinsic definition, Gaussian curvature is in fact an intrinsic property. This point of view was extended to higher-dimensional spaces by Bernhard Riemann (1826-1866) and led to what is known today as Riemannian geometry.



In 1760 Joseph-Luis Lagrange (1736-1813) asked the following question: Given a closed curve in ordinary space, find a surface having the curve as boundary with minimal area. Such a surface is called a minimal surface. In 1776 Jean Baptiste Meusnier (1754-1793) showed the following result: A surface is minimal if and only if its mean curvature vanishes. Minimal surfaces have a simple interpretation in real life: they are the shape a soap film will assume if a wire frame shaped like the curve is dipped into a soap solution and then carefully lifted out.

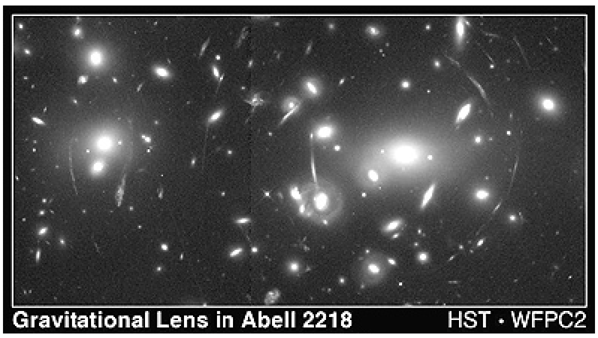
Curves on a surface that minimize distance between the endpoints are called geodesics; they are the shape that an elastic band stretched between the two points would assume. Mathematically they are described using partial differential equations from the calculus of variations. Geodesity of a curve is an intrinsic property.

One way of calculating Gaussian curvature is as the limit of the quotient of the angular excess $\alpha + \beta + \gamma - \pi$ and the area for successively smaller geodesic triangles with angles α,β,γ near the point. Qualitatively, a surface is positively or negatively curved according to the sign of the angular excess for arbitrarily small geodesic triangles. A sphere has everywhere positive curvature since any geodesic triangle on the surface of the earth has angle

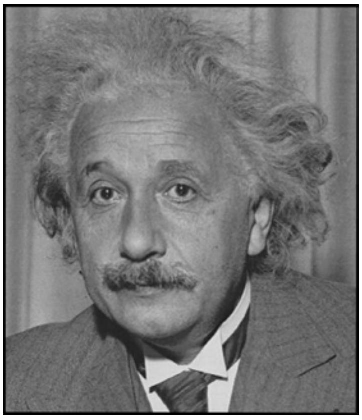


sum greater than π. In the Euclidian case we know from school that the sum of the angels is precisely π. As a Riemannian surface has a specific curvature at each point, we can add up all the curvatures to find the average curvature for the whole surface, called the total curvature of the surface. There is a marvellous theorem, known as the Gauss-Bonnet theorem, which relates this total curvature and a purely topological property of the surface, i.e., quantities that no longer depend on the metric. For a closed surface like a sphere or the surface of a doughnut the Gauss-Bonnet theorem says that the total curvature equals 4π minus 4π times the number of holes in the surface. A sphere does not have any holes, thus the total curvature is 4π. The doughnut has one hole and the total curvature is therefore 0. This interaction between lo-

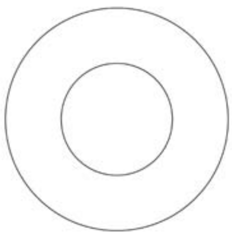
cal concepts like curvature and global properties like the number of holes was the forerunner of many later results in geometry, culminating in the Atiyah-Singer index theorem, for which Michael Atiyah and Isadore Singer were awarded the Abel Prize in 2004.
Now consider manifolds with certain conditions on the curvature. Which possible surfaces do we have if the curvature is assumed to be everywhere zero? Or constant, but nonzero? To answer such questions we are looking for some sort of classification; that the surfaces fulfilling the conditions have certain properties or that they belong to certain classes of surfaces. Mikhail Gromov has published several papers where answers are given to questions relating classification of manifolds with curvature constraints.
Riemannian Geometry also studies higher dimensional spaces. The universe can be described as a three dimensional space. Near the earth, the universe looks roughly like three dimensional Euclidean space. However, near very heavy stars and black holes, the space is curved and bent. The Hubble Telescope has discovered points that have more than one minimal geodesic between them and the point where the telescope is located. This is called gravitational lensing. The amount that space is curved can be estimated by using theorems from Riemannian Geometry and meas-



urements taken by astronomers. Physicists believe that the curvature of space is related to the gravitational field of a star according to a partial differential equation introduced by Albert Einstein (1870-1955). Using results from Riemannian Geometry they can estimate the mass of the star or black hole that causes the gravitational lensing.



As mentioned previously there are many different Riemannian manifolds with the same underlying space, i.e., a manifold can allow different metric structures. A natural problem to address is if it possible, in some natural fashion, to classify all metric spaces? Gromov´s answer to this question is to equip the space of all metric spaces with a metric, now called the Gromov-Hausdorff metric. To measure the distance between two metric spaces they are embedded isometrically into some bigger metric space. As subsets of a common universe we can measure the distance between them; the distance between two compact subsets of a metric space is the minimal value such that for every point in each subset, it is possible to reach some point in the other subset within the range of this minimal distance. The Gromov-Hausdorff distance is the minimal value of all possible embeddings of the two spaces into a third one.



As an example we consider two circles of radius 1 and 2. The Gromov-Hausdorff distance between the two circles is 1, obtained by letting the two circles be concentric. Then the outer circle can be reached within distance 1 from any point of the inner circle, and vice versa.

Around 1980 Gromov published several results concerning this metric space of metric spaces. Two of the most famous theorems bear his name, Gromov´s compactness theorem and Gromov´s convergence theorem.

Symplectic geometry

I 1833 the Irish mathematician William Rowan Hamilton (1805-1865) introduced what is now called Hamiltonian mechanics. It is a reformulation of classical mechanics, motivated from a previous reformulation by Lagrange from 1788. Lagrange formulated classical mechanics through solutions of certain second-order differential equations. Hamilton changed the formalism by considering two sets of coordinates: position coordinates and momentum coordinates. Lagrange´s second-order constraints on an n-dimensional coordinate space now became first-order constraints on a 2n-dimensional phase space. Properties of this particular phase space were extracted and used as a motivation for the definition of symplectic manifolds, technically formulated as manifolds equipped with a certain non-degenerate differential two-form.

There are close relations between symplectic structures and so-called almost complex structures, i.e., answers to the question; is it possible to understand a real 2n-dimensional manifold as an n-dimensional complex manifold? This is a generalisation of what we do when we consider the complex numbers as a real plane, with square root of -1 as the second axis.

For a moment let us consider the famous poohsticks game of Winnie-the-Pooh. The flow of the river can be described by a certain vector field; to each point on the surface the water flow has a direction and a speed. The sticks, which are dropped from the upper side of the bridge, will follow the vector field tracking what is called an integral curve. In the Poohsticks game the aim of Winnie-the-Pooh is to find the fastest track, at least faster than Christopher Robin, Tigger and Eeyore. It is obvious that the stick Winnie-the-Pooh drops in the river will find its way under the bridge. The reason is that the vector field describing the flow has certain nice properties. If we mimic these nice properties, normally described as the Cauchy-Riemann equations, for the symplectic phase space of the Hamiltonian formalism for classical mechanics as given above, we end up by considering maps from the complex numbers into a symplectic manifold satisfying certain properties. Such maps, tracing out a complex curve in the manifold, are called pseudoholomorphic curves or J-curves. They were introduced by Gromov in 1985 and revolutionized the study of symplectic manifolds. In particular they lead to the Gromov-Witten invariants and Floer homology, and play a prominent role in string theory.

Geometric groups

In the citation for this year's Abel Prize, the scientific committee emphasises three different fields where Gromov has played a significant role. Riemannian and symplectic geometry sounds like suitable playgrounds for a world-leading geometer, but what about groups of polynomial growth? What is the connection with geometry and metric spaces?

Have you ever considered how many words our language contains? It is of course not a good idea to start counting words, but nevertheless, let us try. We start by considering words in one single letter, such as "I" and "a" and being rather strict about what we mean by a word, these two seem to be the only ones. The list of words of two letters is much longer, for example "we", "on", "at", “to”, “be”, “go” and “us”, just to mention a few. We are not going to continue this list, but rather change the rules, and focus on a very important mathematical structure. Here are the rules for this mathematical game:

1. Our alphabet contains only two letters, x and y.
2. All combinations of x´s and y´s are words in our language, with two exceptions, the combinations xx and yyy cannot occur as part of a word.

Now let us count the words of this language. We count the legal words by their length, i.e., the number of letters.Denote by W(n) the number of words of length n. An elementary combinatorial argument (which we suppress) tells us that W(n) equals the sum W(n-1)+W(n-5). Thus we can continue the sequence in the rightmost column of the table; 2,3,4,5,7,9,12,16,21,28,37,49 ,65,... This is a sequence of so-called exponential growth, the same phenomena that happens for the world’s total population. It grows fast, but as the population increases, it grows even faster. In this setting the opposite of exponential growth is what we call polynomial growth. Polynomial growth is much slower than exponential growth, e.g., the sequence of all natural numbers 1,2,3,4,5,6,7,... has polynomial growth.

The language in x and y obeying the rules given above is what mathematicians would call the elements of the Projective modular group, PSL(2,Z). What we have shown, or at least indicated, is that this group has exponential growth. Gromov´s theorem from 1981 tells us the following:

Theorem (Gromov, 1981)
A finitely generated group G has polynomial growth if and only if is is virtually nilpotent.

Using this theorem we can now deduce that the projective modular group is not virtually nilpotent. So what? It is not easy to explain what it means for a group to be virtually nilpotent. We have not even explained what is a group. But for the people working in group theory it is very important to know whether a group is virtually nilpotent or not. What we try to communicate is that combining some simple counting and Gromovs theorem; we can say something about PSL(2,Z), one of the most important groups in the modern history of mathematics.
Now, back to our initial question, what does this have to do with geometry? There is in fact a metric, or rather a distance, hidden in this example. There are two very important properties of distance; the triangle inequality and non-de-

Length	Word	W(n)
1	x,y	2
2	xy, yx, yy	3
3	xyx, yyx, yxy, xyy	4
4	xyxy, xyyx, yxyx, yxyy, yyxy	5
5	xyxyx, xyxyy, xyyxy, yx¬yxy, yxyyx, yyxyx, yyxyy	7
6	xyxyxy, xyxyyx, xyyxyx, xyyxyy, yxyxyx, yxyxyy, yxyyxy, yyxyxy, yyxyyx	9

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generacy. The triangle inequality is the generalisation of the truth that the shortest way between two points is the straight line. Non-degeneracy tells us that if the distance between A an B is zero, the A=B. Go back to our language generated by x and y. Given two words we can compose a new word, just by putting the letters together, one word after the other. Remember that xx and yyy is not allowed, if one of those combination appears, we remove it. So sticking together xyx and xyyx gives

xyxxyyx = xyyyx = xx = Ø

(the empty word is always denoted Ø). We define the operation of inverting a word; turn the word around and replace yy by y and vice versa, so that xyyxy is transformed to yyxyx. A small exercise for the reader is now to show that putting together a word and its inverted word gives the empty word. The remarkable fact is that the set of words in this particular alphabet is a metric space, where the distance between two words is the number of letters in the composition of the first word with the inverted second word. This definition satisfies both the triangle inequality and non-degeneracy. Counting points up to a given length is now a perfect analogue of counting points within a ball of given radius around a specific point in the space, namely the empty word. The analogue for a Riemannian surface of counting words of given length is measuring area; words of length less than 1 corresponds to area of a circle of radius 1. Area is a quadratic function of the radius, i.e., "polynomial growth" of degree 2. Similar argument can be used for Riemannian manifolds of any dimension d, counting points is a function of the radius of degree d, i.e., polynomial growth. Gromov´s result can in this context be interpreted as stating that discrete groups, like our alphabet, that are analogues of finite-dimensional manifolds, have a specific group-theoretical characterisation (virtually nilpotent).

Epilogue

Gromovs name is forever attached to deep results and important concepts within Riemannian geometry, symplectic geometry, string theory and group theory. The Abel committee says: "Mikhail Gromov is always in pursuit of new questions and is constantly thinking of new ideas for solutions of old problems. He has produced deep and original work throughout his career and remains remarkably creative. The work of Gromov will continue to be a source of inspiration for many future mathematical discoveries". And as a final remark, let us also quote Dennis Sullivan; "It is incredible what Mikhail Gromov can do, just with the triangle inequality."