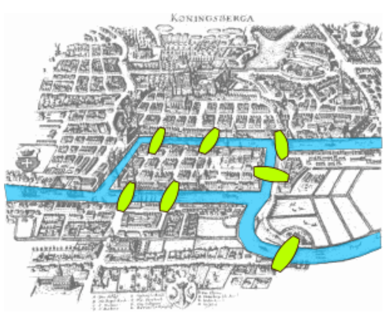
THE ABEL PRIZE 2021

Graph invariants

The city of Königsberg in Prussia (now Kaliningrad, Russia) is located on the banks of the Pregel River. The river surrounds two large islands, Kneiphof and Lomse. The islands are connected to each other, and to the mainland on both sides, by several bridges. All together there are seven bridges in the city. A famous mathematical problem is to find a walk through the city that would cross each of the seven bridges once and only once.



The problem was solved in the 1730´s as the great mathematician Leonhard Euler proved that there can not exist any such walk. But more important than the solution itself, was the fact that Euler, by his way of attacking the problem, gave birth to a new field of mathematics, today known as graph theory.

Graph theory can be viewed as an abstract approach to the study of relations between objects, with graphs as the basic notion.

Definition. A graph $G = (V, E)$ is an ordered pair comprising

- a set of vertices or nodes, V,
- a collection of edges, E, between pairs of nodes.

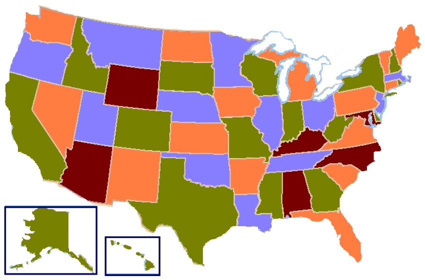
Almost three centuries after Euler walked on the bridges of Königsberg, graph theory has grown as a mathematical discipline and found many important applications. Nevertheless, there are still many unsolved questions and new problems keep popping up.

A basic task in all areas of mathematics is classification. Graph theory is no exception. However, if it is not carved in stone what the classifying invariants of this theory should be. Here are some suggestions.

The first invariant presented is the Chromatic number:

Definition. Let $G = (V, E)$ be a graph. The chromatic number $\gamma(G)$ of G is the least number of colours needed to colour the vertices of the graph so that no two adjacent vertices share the same colour.

The chromatic number is a central concept of the four colour theorem, successfully proven by Appel and Haken in 1976. The four colour theorem says that the chromatic number of a planar graph is four. An equivalent formulation is the four colour map theorem, which states that no more than four colours are needed to colour the regions of a map so that no two adjacent regions have the same colour.



The illustration gives a colouring of the USA, using four colours. Notice that the theorem is not true if we replace the number four by three. You can convince yourself of this fact if you try to colour Bolivia and its neighbouring countries with only three colours.

An important characteristic of a graph is the sparsity, i.e. the number of edges of the graph compared to the maximal number of edges of a graph with the same number of nodes. The extremes are the totally disconnected graph with no edges at all, and the complete graph which includes all possible edges; the number of edges of a complete n-node graph is $\binom{n}{2}$. Let $G = (V, E)$ be a graph, and let $S \subset V$ be any subset of vertices of G. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all the edges in E that have both endpoints in S.

The next invariant we define is the maximal clique size:

Definition. Let G be a graph. A clique in G is a complete induced subgraph. The size of the largest clique in G is denoted $\omega(G)$.

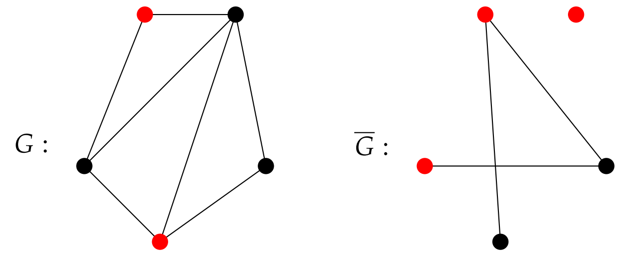
In a complete graph, all nodes are adjacent to each other and in a colouring they must all have different colours. Thus the chromatic number of a complete graph equals the number of vertices. In an arbitrary graph the chromatic number may exceed the maximal clique size, i.e. the equality reduces to an inequality $\omega(G) \leq \gamma(G)$.

Definition. A graph $G = (V, E)$ is said to be perfect if for all induced subgraphs $S \subset G$ the chromatic number and the clique size coincides, i.e. $\omega(G[S]) = \gamma(G[S])$.

In a paper published in 1972 Abel Prize Laureate László Lovász proved the Perfect graph theorem:

Theorem. A graph $G = (V, E)$ is perfect if and only if the complement graph $\overline{G}$ is perfect.

The complement graph $\overline{G}$ has the same nodes as G, but the edge set is complementary, i.e. if $e \in E(G)$ then $e \notin E(\overline{G})$ and vice versa. The following example illustrates the perfect graph theorem:



$\gamma(G) = \omega(G) = 3$ $\qquad$ $\gamma(\overline{G}) = \omega(\overline{G}) = 2$

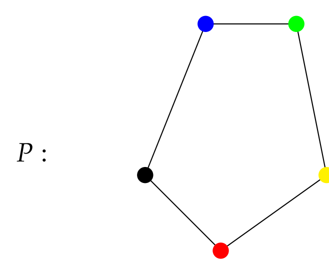
The next graph invariant we emphasize is the independence number of the graph. An independent subset of a graph is a set of pairwise non-adjacent nodes.

Definition. The independence number of a graph $G(V, E)$, denoted $\alpha(G)$, is the cardinality of the largest independent subset of G.

In the above illustration the sets of red nodes in the two graphs are independent of maximal size, thus we have $\alpha(G) = 2$, $\alpha(\overline{G}) = 3$. The invariants of the above illustration reflects a general fact; that the maximal clique number of a graph coincides with the independence number of the complement graph, $\omega(G) = \alpha(\overline{G})$.

During the late 1920´s, the electronic experts Harry Nyquist and Ralph Hartley introduced some fundamental ideas related to the transmission of information, particularly in the context of the telegraph as a communications system. Some years later, during the 1940´s, Claude Shannon developed the concept of channel capacity, based in part on the ideas of Nyquist and Hartley, and then formulated a complete theory of transmission of information.

The Shannon capacity models the amount of information that can be transmitted across a noisy communication channel in which certain signal values can be confused with each other. The confusion is encoded in the confusion graph, where the different signals are represented as nodes, and a possible confusion between two signals is represented by an edge between the two signals.



The figure shows a confusion graph, illustrated by a pentagon graph.
The red signal may be confused with the black and the yellow, but will be distinguished from the blue and the

green, and so on. The Shannon capacity of the graph is at least 2, e.g. represented by the independent set {•, •}. If we instead of single signals decide to transmit pairs of signals, then in fact we can find five pairs which cannot be confused, e.g.

(••) (••) (••) (••) (••)

The Shannon capacity is measured per signal, thus using pairs we increase the capacity to $\sqrt{5}$. For a long time it was unknown whether it was possible to increase the capacity even more by using more complex signal combinations. The mathematical way of addressing this problem goes as follows:

Definition. Let $G = (V, E)$ and $H = (W, F)$ be two graphs. The strong graph product of G and H, denoted $G * H$ is the following graph:

i) The vertices of $G * H$ is the set of all pairs (g, h) with $g \in V$ and $h \in W$.

ii) There is an edge between (g, h) and (g', h') if either of

   (a) $g = g'$ and $[h, h'] \in F$
   (b) $h = h'$ and $[g, g'] \in E$
   (c) $[g, g'] \in E$ and $[h, h'] \in F$

   is satisfied.

Iterating the product construction gives a multi-product graph $G_1 * G_2 * \cdots * G_k$.

With this definition in hand we can formally introduce the Shannon capacity, as was done by Claude Shannon in the 1940´s.

Definition. Let $G = (V, E)$ be a graph and k a natural number. The Shannon capacity of G, denoted $\Theta(G)$ is defined as

$$\Theta(G) = \lim_{k\to\infty} (\alpha(G * G * \cdots * G))^{\frac{1}{k}}$$

where the product is the k-fold product of copies of G.

The Shannon capacity of an arbitrary graph is not easily computed and the computational complexity is not determined. As pointed out above, the Shannon capacity of the pentagon graph remained unknown for many years. It was finally found by Lovász in 1979. As a main tool Lovász introduced another graph invariant, known as the Lovász number.

Definition. Let $G = (V, E)$ be a graph on n vertices. An ordered set of n unit vectors $U = \{u_i \,|\, i \in V\} \subset \mathbb{R}^N$ is called an orthonormal representation of G in $\mathbb{R}^N$, if u_i and u_j are orthogonal whenever vertices i and j are not adjacent in G, i.e.

$$u_i^T u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } [i, j] \notin E \end{cases}$$

Definition. The Lovász number of the graph $G(V, E)$ is defined as follows:

$$\theta(G) = \min_{c,U} \max_{i\in V} \frac{1}{(c^T u_i)^2}$$

where c is a unit vector in $\mathbb{R}^N$ and U is an orthonormal representation of G in $\mathbb{R}^N$.

The Lovász number $\theta(G)$ can be computed numerically to high accuracy in polynomial time. Lovász proved the so-called Lovász sandwich theorem which relates the maximal clique number, the Lovász number and the chromatic number.

Theorem. For a graph $G = (V, E)$ we have the inequalities

$$\omega(G) \le \theta(\overline{G}) \le \gamma(G)$$

Lovász also proved that the Lovász number is an upper bound for the Shannon capacity, i.e.

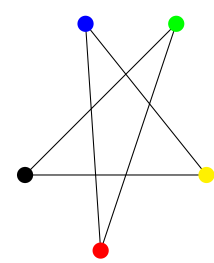
$$\alpha(G) \le \Theta(G) \le \theta(G)$$

where the leftmost inequality is more or less obvious. Combining the two inequalities and using the fact that $\alpha(G) = \omega(\overline{G})$ we get

$$\omega(\overline{G}) \le \Theta(G) \le \theta(G) \le \gamma(\overline{G})$$

It follows that for a perfect graph G, where $\omega(G) = \gamma(G)$ the above inequalities are equalities and the Shannon capacity equals the Lovász number.

So what about the pentagon graph P? The complement graph $\overline{P}$:



This graph has chromatic number $\gamma(\overline{P}) = 3$ and maximal clique number $\omega(\overline{P}) = 2$. Thus we have

$$2 \le \Theta(G) \le \theta(G) \le 3$$

We have already seen that $\sqrt{\alpha(P * P)} \ge \sqrt{5}$, narrowing down the interval to

$$\sqrt{5} \le \Theta(G) \le \theta(G) \le 3$$

Finally, we show that the Lovász number for the pentagon graph, $\theta(P) \le \sqrt{5}$, and it follows that

$$\Theta(G) = \theta(G) = \sqrt{5}$$

In fact, let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of unit vectors in $\mathbb{R}^3$ given by

$$u_k = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\beta_k \\ \sin\alpha\sin\beta_k \end{pmatrix}$$

where $\cos\alpha = \frac{1}{\sqrt[4]{5}}$ and $\beta_k = \frac{2\pi k}{5}$. Then we have

$$\begin{aligned} u_k \cdot u_\ell &= \cos^2\alpha + \sin^2\alpha(\cos\beta_k\cos\beta_\ell + \sin\beta_k\sin\beta_\ell) \\ &= \cos^2\alpha + \sin^2\alpha\cos(\beta_k - \beta_\ell) \\ &= \frac{1}{\sqrt{5}} + (1 - \frac{1}{\sqrt{5}})\cos(\beta_k - \beta_\ell) \end{aligned}$$

If $|k - \ell| = 2$ we have $\frac{2k\pi}{5} - \frac{2\ell\pi}{5} = \frac{4\pi}{5}$ and $\cos\frac{4\pi}{5} = -\frac{1}{\sqrt{5}-1}$. Thus

$$u_k \cdot u_\ell = \frac{1}{\sqrt{5}} + (1 - \frac{1}{\sqrt{5}})\frac{-1}{\sqrt{5}-1} = 0$$

and the ordered set of 5 unit vectors $U = \{u_i \mid i \in V\} \subset \mathbb{R}^3$ constitutes an orthonormal representation of P in $\mathbb{R}^3$. Let $c = (1, 0, 0)$. The Lovász number of P is then

$$\theta(P) = \min_{c,U} \max_{i \in V} \frac{1}{(c^T u_i)^2} \leq \max_{i \in V} \frac{1}{(c^T u_i)^2} = \frac{1}{(\frac{1}{\sqrt[4]{5}})^2} = \sqrt{5}$$