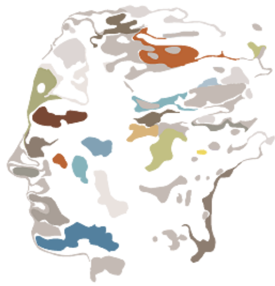
Abel Prize 2009



Groups and growth

Have you ever considered how many words our language contains? The topic of this note is to give an answer to a somewhat simplified version of this complete stupid task, and relate this to Gromovs paper from 1981, about growth of virtually nilpotent groups.

It is of course no good idea to start counting words, but nevertheless, et us try. We start by considering words in one single letter, such as I and a. Being rather strict about what we mean by a word, these two seem to be the only ones. The list of words of two letters is much longer, for example we, on, at, to, be, go and us, just to mention a few. We are not going to continue this list, but rather change the rules, and focus on a very important mathematical structure. Here are the rules for this mathematical game:

1. Our alphabet contains only two letters, x and y.
2. All combinations of x´s and y´s are words in our language, with two exceptions, the combinations xx and yyy are not accepted.

Now let us count the words of this language. We list the legal words by their length, i.e. the number of letters.

Length	Words	Number
1	x,y	2
2	xy, yx, yy	3
3	xyx, yyx, yxy, xyy	4
4	xyxy, xyyx, yxyx, yxyy, yyxy	5
5	xyxyx, xyxyy, xyyxy, yxyxy, yxyyx, yyxyx, yyxyy	7
6	xyxyxy, xyxyyx, xyyxyx, xyyxyy, yxyxyx, yxyxyy, yxyyxy, yyxyxy, yyxyyx	9

Denote by W(n) the number of words of length n. An elementary combinatorial argument (which we suppress) tells us that W(n) equals the sum W(n-1)+W(n-5). Thus we can continue the sequence in the rightmost column of the table; 2,3,4,5,7,9,12,16,21,28,37,49,65,... This is a sequence of socalled exponential growth, the same phenomena thaht happens for the world´s total population. It grows fast, but as the population increases, it grows even faster. In this setting the contrary to exponential growth is what we call polynomial growth. Polynomial growth is much slower than exponential growth, e.g. the sequence of all natural numbers 1,2,3,4,5,6,7,... has polynomial growth.

The language in x and y obeying the rules given above, is what mathematicians would call the elements of the ***Projective modular group, PSL(2,Z).*** What we have shown, or at least indicated, is that this group has exponential growth. Gromov´s theorem from 1981 tells us the following:

Theorem (Gromov, 1981)
A finitely generated group G has polynomial growth if and only if is is virtually nilpotent.

Using this theorem we can now deduce that the projective modular group is not virtually nilpotent. So what? It is not easy to explain what it means for a group to be virtually nilpotent. We have not even explained what a groupis . But for the people working in group theory it is very important to know wether a group is virtually nilpotent. What we try to communicate is that combining some simple counting and Gromovs theorem, we can say something about PSL(2,Z), one of the most important groups in the modern history of mathematics.