# 2004

# Sir Michael Atiyah
# and
# Isadore M. Singer

ABEL
PRISEN

# Autobiography

**Sir Michael Atiyah**

I was born in London on $22^{nd}$ April 1929, but in fact I lived most of my childhood in the Middle East. My father was Lebanese but he had an English education, culminating in three years at Oxford University where he met my mother, who came from a Scottish family. Both my parents were from middle class professional families, one grandfather being a minister of the church in Yorkshire and the other a doctor in Khartoum.

My father worked as a civil servant in Khartoum until 1945 when we all moved permanently to England and my father became an author and was involved in representing the Palestinian cause. During the war, after elementary schooling in Khartoum, I went to Victoria College in Cairo and (subsequently) Alexandria. This was an English boarding school with a very cosmopolitan population. I remember priding myself on being able to count to 10 in a dozen different languages, a knowledge acquired from my fellow students.

At Victoria College I got a good basic education but had to adapt to being two years younger than most others in my class. I survived by helping bigger boys with their homework and so was protected by them from the inevitable bullying of a boarding school.

In my final year, at the age of 15, I focused on mathematics and chemistry but my attraction to colourful experiments in the laboratory in due course was subdued by the large tomes which we were expected to study. I found memorizing large bodies of factual information very boring and so I gravitated towards mathematics where only principles and basic ideas matter. From this point on it seemed clear that my future lay in mathematics.

There were vague allusions to some of my older Lebanese relatives having shown mathematical talent and one of my maternal uncles had been a brilliant classical scholar, ending up as a Fellow of an Oxford college. Classics and mathematics were

M. Atiyah (✉)
School of Mathematics, University of Edinburgh, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland, UK
e-mail: M.Atiyah@ed.ac.uk

H. Holden, R. Piene (eds.), *The Abel Prize*, 
DOI 10.1007/978-3-642-01373-7_5, 

Aged 2, known as "the abbott"

the two traditional subject studied by serious scholars in England in former years, so I may have inherited some mathematical potential from both sides of my family. My younger brother Patrick became a distinguished law professor and there is a clear affinity between the legal and mathematical minds, both requiring the ability to think clearly and precisely according to prescribed rules. One of my mathematical contemporaries, and a close friend, demonstrated this by entering the legal profession and ending up as Lord Chancellor of England.

In England I completed my school education by being sent to Manchester Grammar School (MGS), widely regarded as the leading school for mathematics in the country. Here I found that I had to work very hard to keep up with the class and the competition was stiff. We had an old-fashioned but inspiring teacher who had graduated from Oxford in 1912 and from him I acquired a love of projective geometry, with its elegant synthetic proofs, which has never left me. I became, and remained, primarily a geometer though that word has been reinterpreted in different ways at different levels. I was also introduced to Hamilton's work on quarternions, whose beauty fascinated me, and still does. I have been delighted in the way that quarternions have enjoyed a new lease of life in recent years, underlying many exciting developments.

At MGS the mathematics class, a small and highly selected group, were all trained for the Cambridge scholarship examinations. In due course all the class went on to Cambridge except for one who went instead to Oxford. The students were steered to different colleges, depending on their abilities, and I was one of the top three sent to Trinity College, home of Isaac Newton, James Clerk Maxwell, Bertrand Russell and other famous names.

Having got my Trinity scholarship in 1947 I had the choice of going straight to Cambridge or of postponing my entry until I had done my two-year stint of National Service. I chose the latter for the vague idealistic reason that I should do my duty and not try to escape it (as many of my contemporaries did) by running away to university with the hope of indefinite postponement of military service. One should remember that this was just two years after the end of the war and my age group was one of the first to have escaped the harsh reality of war-time service.

In fact my military career was something of an anti-climax. I served as a clerical officer in a very routine headquarters. There were some advantages—I kept myself physically fit and even cycled home every weekend. I met a wide cross-section of humanity at a formative stage of my life and I was removed from the competitive hot-house atmosphere of mathematical competition. In my spare time I read mathematics for my own enjoyment—I remember enjoying Hardy and Wright's book on Number Theory and I even read a few articles in the Encyclopaedia where I first encountered the ideas of group theory.

The normal length of National Service at the time was two years but my tutor at Cambridge managed to persuade the authorities that I should be allowed out a few months early to attend the "Long-Vacation Term". This was the period in the summer when those with extensive practical work, such as engineers, came up for additional courses. There was no requirement for mathematics students to spend the summer in Cambridge, but scholars of the college could opt to do so at little cost, and I remember that period as quite idyllic. I enjoyed the beauty of Cambridge, played tennis and studied on my own at a leisurely pace. It all helped make the readjustment to civilian life smooth and pleasant and it gave me a head start when the academic year began in earnest.

Trinity, because of its reputation, attracted a large number of exceptionally talented mathematics students. The competition among us was friendly but fierce and it was not clear until the end of the final year where one stood in the pecking order and what ones chances of a professional mathematical career would be. In fact I came top of the whole university in the crucial examination and this gave me the confidence to plan ahead.

By the time I started in Cambridge in 1949 I was 20. Instead of being two years younger than others I was two years older (although many others had also done National Service). The additional maturity was an advantage, even if I seemed to have lost two valuable years.

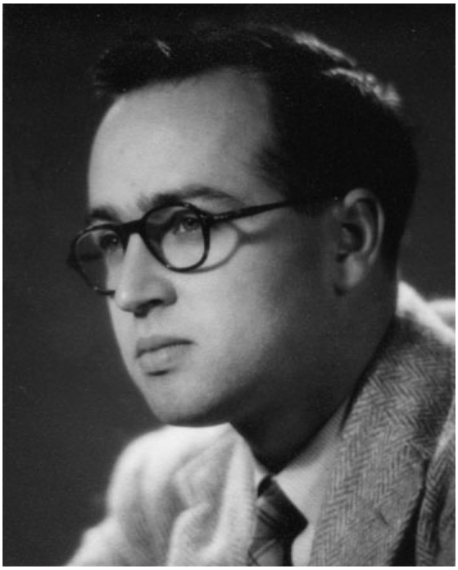
After my first degree I had to make a crucial choice in picking my research supervisor. Here I made the right decision, opting for Sir William Hodge as the most famous mathematics professor in the field of geometry. He it was who steered me into the area between algebraic and differential geometry where he had himself made his name with his famous theory of "Harmonic Integrals". Although the war had interfered with his career he was still in touch with new developments and he also had wide international contacts.

When I started research the geometrical world was undergoing a revolution based on the theory of sheaves and the topological underpinning of the theory of characteristic classes. Here the leading lights were the young post-war generation, just a few years older than me. Jean-Pierre Serre and Friedrich Hirzebruch were two of these whose influence on me was decisive, and I also met them very early.

My thesis grew out of this area and my own background in classical projective geometry. There were two parts, one dealing with vector bundles on algebraic curves (which later became a very popular topic) and the other, jointly with Hodge, on "Integrals of the Second Kind". This was a modern treatment of an old subject.

My thesis in 1954 earned me one of the highly-sought Research Fellowships at Trinity, which are safe predictors of future academic success. By this time I needed larger horizons and with Hodge's help and encouragement I got a Fellowship to go to the Institute for Advanced Study in Princeton. This I did in 1955 just after

Graduation, 1952, with friends. From left to right: James Mackay Lord Chancellor, MFA, Ian Macdonald FRS mathematician, John Polkinghorne FRS physicist and theologian, John Aitcheson professor of statistics



Trinity Fellowship, 1954

Wedding, 1955

marrying Lily Brown, a fellow student, who had come down from Edinburgh to do a Ph.D. under Mary Cartwright. She already had a university position in London but, in those days, it was customary to put the husband's career first, so she resigned her post and came with me to Princeton. A few years later such a sacrifice would not have been expected, though there is never an easy solution.

Princeton was a very important part of my mathematical development. Here I met many of those who would influence or collaborate with me in the future. In addition to Serre and Hirzebruch there were Kodaira and Spencer, of the older geometers, and Bott and Singer of the younger ones. In later times I came frequently to the Institute while on sabbatical leave and for three years, 1969–72, I was a professor on the Faculty. It was my second mathematical home.

My subsequent career oscillated between Cambridge, Oxford and Princeton. In 1957 I returned to Cambridge as a University Lecturer and in 1961 I moved to Oxford, first as a Reader and then from 1963–69 as Savilian Professor of Geometry. After my stay at Princeton from 1969–72 I returned to Oxford as a Royal Society Research Professor, staying in that post until in 1990 I moved back to Cambridge as Master of Trinity College.

If these were the universities where I had permanent positions an important part was also played by Harvard and Bonn. During my close collaborations with Bott and Singer I spent two sabbatical terms at Harvard and for around twenty-five years I used to go to Bonn for the annual Arbeitstagung. These were enormously exciting events with many new results being discussed and with a stellar cast of participants.

During the early years in Bonn much attention was paid to Hirzebruch's Riemann–Roch Theorem and its subsequent generalization by Grothendieck. Almost at the same time Bott made his famous discovery of the periodicity theorem in the classical groups. By being around at the right time, having the right friends and playing around with the formulae that emerged, I soon realised the close links between the work of Bott and Grothendieck. This led to new concrete results in algebraic topology which convinced me that it would be worth developing a topological version of Grothendieck's $K$-theory. This grew into a significant enterprise and it was natural that Hirzebruch should join me in developing it. He had more experience of Lie groups and their characteristic classes and his own earlier work tied in with my developing interest.

Over the subsequent years Hirzebruch and I wrote many joint papers on various aspects and applications of $K$-theory. It was an exciting collaboration from which I learnt much, not least in how to write papers and present lectures. He was, in effect, an elder brother who continued my education.

Some of the remarkable consequences of Hirzebruch's Riemann–Roch Theorem had been the integrality of various expressions in characteristic classes. A priori, since these formulae had denominators, the answers were rational numbers. In fact, under appropriate hypotheses, they turned out to be integers. For complex algebraic manifolds this followed from their interpretation as holomorphic Euler characteristics, a consequence of the Riemann–Roch Theorem. For other manifolds Hirzebruch had been able to deduce integrality by various topological tricks, but this seemed unsatisfactory. Topological $K$-theory gave a better explanation for these integrality theorems, closer to the analytic proofs derived from sheaf theory in the case of complex manifolds.

A particularly striking case was the fact that an expression called by Hirzebruch the $\hat{A}$-genus was an integer for spin-manifolds. It was the attempt to understand this fact that eventually led Singer and me to our index theorem. Because of the comparison with analytic methods on complex manifolds it was natural to ask if there was any analytical counterpart for spin-manifolds.

A key breakthrough came with the realization that Dirac had, thirty years before, introduced the famous differential operator that bears his name. Singer, with a better background in physics and differential geometry, saw that, on a spin-manifold, one could, using a Riemannian metric, define a Dirac operator acting naturally on spinor fields. From my apprenticeship with Hirzebruch I was familiar with the character formulae for the spin representations and so it was easy to see that the index of the Dirac operator should be equal to the mysterious $\hat{A}$-genus.

All this started while Singer was spending a sabbatical term with me in Oxford. We also had a brief visit from Stephen Smale, just returned from Moscow, who told us that Gel'fand had proposed the general problem of computing the index of any elliptic differential operator.

Because of our knowledge of $K$-theory we saw that the Dirac operator was in fact the primordial elliptic operator and that, in a sense, it generated all others. Thus a proof of the conjectured index formula for the Dirac operator would yield a formula for all elliptic operators.

In retrospect it might seem surprising that the Dirac operator had not been seriously studied by differential geometers before our time. Nowadays it all seems transparently obvious to a first-year graduate student. But the reasons for this neglect of the Dirac operator are not far to seek. First the Dirac equation in space-time is hyperbolic, not elliptic, second, spinors are mysterious objects and, unlike differential forms, have no natural geometric interpretation. The first point is analogous to the difference between Maxwell's equation and Hodge theory, and it took nearly a century for this gap to be bridged. The mysterious nature of spinors is an additional reason and so a delay of thirty years is quite modest.

The road that Singer and I took to arrive at the index theorem was that of a solution looking for a problem. We knew the precise shape of the answer, but the answer to what? Such an inverse approach may not be unique but it is certainly unusual.

Having formulated our index theorem, Singer and I had to search hard for a proof. Here our many good friends in the analytical community were invaluable, and we had to master many new techniques. This was easier for Singer since his background was in functional analysis.

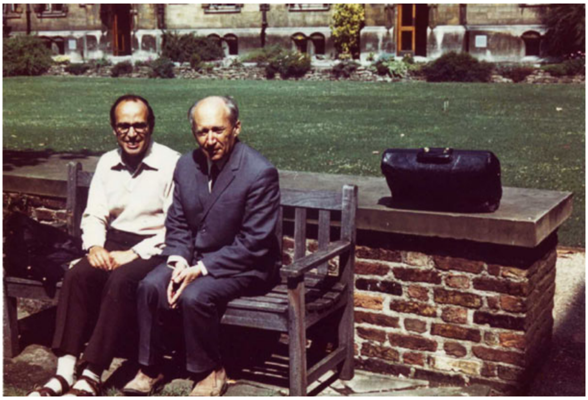
Over the subsequent decades the index theorem in its various forms and generalizations occupied most of our efforts. A particularly interesting strand was the Lefschetz fixed point formula which I developed with Raoul Bott, and the fuller understanding of elliptic boundary value problems which was also joint with Bott. It was during this period that I spent two sabbatical terms at Harvard and I recall this as a particularly stimulating and fruitful time.

Another important extension of the index theorem which required the collective efforts of Bott, Singer, Patodi and myself was the local form of the index theorem and the contribution of the boundary arising from the $\eta$-invariant. This was a spectral invariant, analogous to the $L$-functions of number theory and originating in fact in a beautiful conjecture of Hirzebruch on the cusps of Hilbert modular surfaces. Most of this work was done while I was a professor at Princeton, with my collaborators as visitors.

Graeme Segal, who was one of my early research students, collaborated on the equivariant version of the index theorem as well as on aspects of $K$-theory.

In 1973 I returned to Oxford and while I had no formal teaching duties I acquired, over the years, a string of talented research students who also influenced my research. Nigel Hitchin moved to Princeton with me and then returned to Oxford and we collaborated on many topics. In 1973 I also met up again with my Cambridge contemporary Roger Penrose who had now become a theoretical physicist. We interacted fruitfully as soon as we realized that the complicated contour integrals arising in his twistor theory could be reinterpreted in terms of sheaf cohomology. This established a key bridge between his group and mine.

In due course this led on to the study of instantons and monopoles and opened doors to a wider physics community. It also led to the spectacular results of Simon



With I.M. Gelfand in Oxford, 1973

The Queen opening a new building at Trinity College, 1993

Donaldson on 4-dimensional geometry, one of the highlights of 20$^{\text{th}}$ century mathematics.

By the late 70's the interaction between geometry and physics had expanded considerably. The index theorem became standard form for physicists working in quantum field theory, and topology was increasingly recognized as an important ingredient. Magnetic monopoles were one manifestation of this, as I had learnt from Peter Goddard. I was fortunate to get to know Edward Witten fairly early in his career while he was a Junior Fellow at Harvard. For over thirty years he has been recognised as the driving force among theoretical physicists exploring the frontiers of their subject. I learned a great deal from him and he has provided mathematicians with an entrée to theoretical physics which is remarkable in its richness and sophistication. The influence of new ideas in quantum field theory and string theory has been widespread and much more may be in store.

For most of my working life I have held research posts which left me free to concentrate on my own work. So, later in life, I felt I had a duty to take on various administrative roles which are in any case more suited to grey hairs. I have presided over the London Mathematical Society, The Royal Society, Trinity College and the Newton Institute, while currently I am President of the Royal Society of Edinburgh. I have also served on the advisory committees of mathematical institutes in many countries.

In my youth I received the Fields Medal (1966) and in old age the Abel Prize (2004). I have been fortunate in many things, the quality of my collaboration and students, the support of many centres of research and a firm family base.

# Autobiography

**Isadore M. Singer**

My mother, Freda Rosemaity, and father, Simon Singer, were born in Poland. After World War I, they immigrated to Toronto, Canada where they met and were married. My father was a printer and my mother a seamstress. In the early 1920's they moved to Detroit, Michigan. I was born there in 1924.

My parents struggled through the depression. In the mid 1930's we were able to move from a poor neighborhood to a better one with a good school system. I was an all A student who did not find my courses challenging. In the summers I played baseball during the day and read novels at night.

The periodic table, explained by my high school chemistry teacher, awakened me to science. I was enthralled by its symmetry and began devouring popular books on chemistry and physics. In my senior year I became president of the Science Club and lectured on Relativity to club members.

I won a tuition scholarship to the University of Michigan and moved to Ann Arbor in September 1941. Three months later the United States entered World War II. I enlisted in the Signal Corps; it promised not to induct me into active service until I received my Bachelor of Science degree. Nevertheless I rushed through college, graduating in January 1944. I majored in physics but still regret that I did not take advantage of the superb mathematics faculty at Michigan.

The two physics courses that intrigued and puzzled me were Relativity and Quantum Mechanics. I decided I needed a better mathematical background and was determined to get it while on active duty. When the war ended I was in charge of a Signal Corps school in the mudflats of Luzon for the Phillippino Army. Fortunately, the University of Chicago offered correspondence courses in classical Differential Geometry and in Group Theory. My evenings were spent working problems while my comrades played poker. Mail call brought letters from my family and corrected problem sets from Chicago.

I.M. Singer (✉)
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA
e-mail: ims@math.mit.edu

H. Holden, R. Piene (eds.), *The Abel Prize*,
DOI 10.1007/978-3-642-01373-7_6, 

Central High School Graduates, 1941: Shoshana, Ben, Yitzchak, Miriam, Lillit

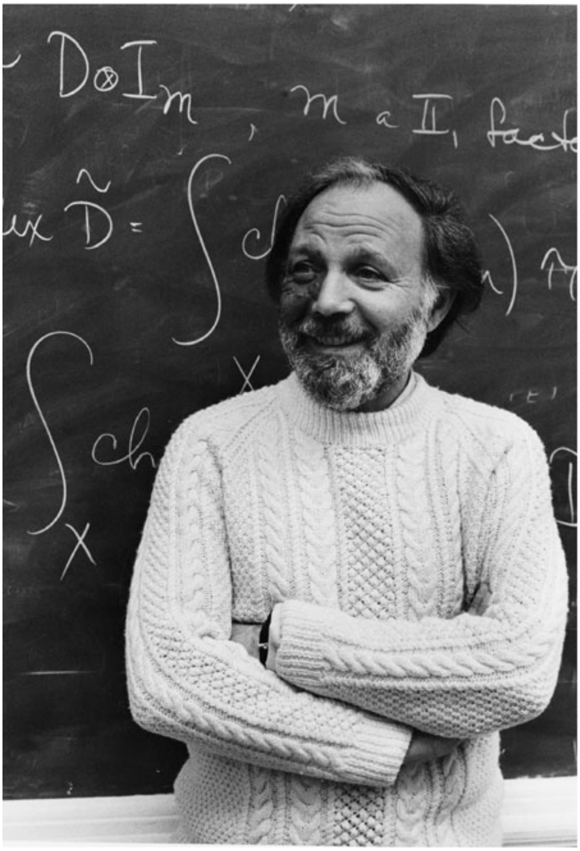
When I came home at the end of 1946, I was admitted to the graduate program in the mathematics department at the University of Chicago. I had planned to return to physics after a year, but mathematics was so elegant and exciting to learn that I stayed put. I specialized in analysis under the direction of I.E. Segal. S.S. Chern who joined the mathematics department during my last year, taught a fascinating course in Differential Geometry that described the Global Geometry of fiber bundles in terms of differential forms.

After receiving my Ph.D. in June 1950, I moved to MIT for two memorable years as an Instructor. We organized many seminars to learn the remarkable postwar developments in topology, analysis, and geometry. And it was there that I met Warren Ambrose who would become a longtime friend and collaborator. He asked me to explain Chern's course; we spent many nights drinking coffee and driving around Boston discussing geometry.

After assistant professorships at UCLA and Columbia University, I was a fellow at the Institute for Advanced Study in 1955, a very special year. I met and talked with mathematicians who were or became famous: Michael Atiyah, Raoul Bott, Fritz Hirzebruch, J.-P. Serre, and my dear graduate school friend Arnold Shapiro.

When I returned to MIT in 1956, Ambrose and I, with the help of our students, modernized and extended Chern's approach to Global Differential Geometry. We also revised many undergraduate courses to bring them up to date with the postwar advances in mathematics. It was an exciting time. Students caught our enthusiasm and instructors brought new perspectives from other places. My small office was crowded with people as I carried out simultaneous discussions on different topics, on holonomy, on commutative Banach algebras, on Eilenberg–MacLane spaces.

The Sloan Foundation awarded me a fellowship for the academic year 1961–62. I spent the fall on the Isle of Capri reviewing a manuscript on the Infinite Groups of E. Cartan, joint work with Shlomo Sternberg [21]. In December I called Michael Atiyah and asked if there was room for me at Oxford. He simply said, "Come", though he had only arrived a few months earlier himself. I came in January; Michael



At MIT

Singer and Atiyah

was a most hospitable colleague. Our collaboration of more than twenty years started then. We conjectured in the spring that the $\hat{A}$ genus of a spin manifold was the index of the Dirac operator [a generalization of Dirac's equation for the spinning

With family

electron] on that Riemannian spin manifold. We quickly extended the conjecture to include geometric elliptic differential operators and found a proof in the fall of 1962. Gelfand's insight[1] and the consequences of Seeley's MIT thesis[2] allowed us to further generalize our result, giving a topological formula for the index of any elliptic operator on a compact manifold [19]. The index theorem and its proof brought together analysis, geometry, and topology in unexpected ways. We extended it in different directions over a period of fifteen years.

My collaboration with Sir Michael has been a major part of my mathematical work. His expertise in topology and algebraic geometry and mine in analysis and differential geometry made a good match. Working with him was exciting and fun. All ideas were worth exploring at the blackboard, erased if nonsense, pursued intensively otherwise.

Our last collaboration (to date) in 1984 applied the families index theorem to the computation of chiral anomalies in gauge theory and string theory [67]. In the mid 1970's mathematicians realized that gauge fields in physics, which describe the interactions of fundamental particles, were the same as connections on principal bundles. Computing the dimension of the moduli of self dual gauge fields was an early application of the index theorem [54]. Our 1984 paper encouraged high energy theorists to apply the families' index theorem and its $K$-theory formulation to problems in string theory.

When I came to the University of California in Berkeley in 1977, I started a math/physics seminar. I wanted to know how to quantize gauge fields and why

[1] I.M. Gelfand, *On elliptic equations*, Russian Math. Surveys (1960) no 3, 113.

[2] R.T. Seeley, *Singular integral operators on compact manifolds*, Amer. J. Math. (81) 1999 658–690.

With I.M. Gelfand in Oxford, 1953

self dual fields were important in physics. Three gifted students, Dan Freed, Daniel Friedan, and John Lott helped me run the seminar. Daniel taught me much about quantum field theory on long walks at physics workshops. Orlando Alvarez joined the physics department in 1982 and became an enthusiastic participant in the seminar. We have been working together for almost fifteen years. M.J. Hopkins suggested that our last paper [102] will have some applications to elliptic cohomology. When I don't understand some physics, I call Orlando for an explanation.

I brought the seminar with me when I returned to MIT in 1984. It still flourishes. Last year was devoted to the paper by Kapustin and Witten[3] building a bridge between Electromagnetic Duality and the Geometric Langlands program in representation theory and number theory. That $S$-duality in string theory may impact number theory and/or vice versa is an exciting prospect.

Most of my academic life has been at MIT, a very fertile environment for me. My collaboration with colleagues here produced interesting mathematics—twenty papers starting with Warren Ambrose on Holonomy in 1953 [3]. My ongoing research with Richard Melrose and V. Mathai extends the index theorem to the case of twisted $K$-theory and the case where the manifold has no spin$_C$ structure [104, 106, 108].

I'm grateful to MIT for allowing me to teach what I want, the way I want, and for giving me ample time to do my own work. It has also been enthusiastic about my activities in Washington DC in support of science, a period that lasted thirty

[3] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, arXiv:hep-th/0604151.

I.M. Singer (Courtesy of the MIT News Office)

years. Most interesting were membership in the White House Science Council during the Reagan administration and chairmanship of the National Academy of Sciences' "Committee on Science and Public Policy".

In 1988 J.P. Bourguignon and J.L. Gervais arranged for my appointment as Chair of Geometry and Physics, Foundation of France. I gave a weekly seminar on 'Introduction to String Theory for Mathematicians' at the École Normale and École Polytechnique. The mixture of mathematicians and physicists made a responsive audience. I fondly remember Claude Itzykson gently asked me leading questions. We became good friends. Laurent Baulieu was also a member of the audience. We talked, and soon wrote the first of several papers on cohomological quantum field theories [76, 79, 84]. I believe my seminar opened new lines of communication between different Laboratories in and around Paris.

In 1982, S.S. Chern, Calvin Moore, and I founded the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, funded by the National Science Foundation. MSRI will soon celebrate its twenty-fifth anniversary with a conference reviewing its past successes. A major theme of the conference is the recognition and support of new mathematics and its applications.

Anticipating new directions is not easy. The growth and evolution of mathematics since I became a graduate student sixty years ago is astonishing. To have been a key participant in the development of index theory and its applications to physics is most gratifying. And beyond that I am fortunate to have experienced first hand the impact of ideas from high energy theoretical physics on many branches of mathematics.

# The Atiyah–Singer Index Theorem

**Nigel Hitchin**

## 1 Introduction

The Abel Prize citation for Michael Atiyah and Isadore Singer reads: "The Atiyah–Singer index theorem is one of the great landmarks of twentieth-century mathematics, influencing profoundly many of the most important later developments in topology, differential geometry and quantum field theory". This article is an attempt to describe the theorem, where it came from, its different manifestations and a collection of applications. It is clear from the citation that the theorem spans many areas. I have attempted to define in the text the most important concepts but inevitably a certain level of sophistication is needed to appreciate all of them. In the applications I have tried to indicate how one can use the theorem as a tool in a concrete fashion without necessarily retreating into the details of proof. This reflects my own appreciation of the theorem in its various forms as part of the user community. The vision and intuition that went into its proof is still a remarkable achievement and the Abel Prize is a true recognition of that fact.

## 2 Background

**2.1. The Index.** If $A : V \to V$ is a linear transformation of finite dimensional vector spaces, then as every undergraduate knows, $\dim\ker A + \dim\operatorname{im} A = \dim V$, so the dimension of the kernel of $A$ and its cokernel $V/\operatorname{im} A$ are the same. In infinite dimensions this is not true. Of course, if $V$ is a Hilbert space, then $\ker A$ may not be finite dimensional anyway, but we can restrict to the class of operators called *Fredholm* operators—bounded operators with finite-dimensional kernel, closed image and finite-dimensional cokernel. In this case the *index* is defined as

$$\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A.$$

N. Hitchin (✉)
Mathematical Institute, Oxford University, 24–29 St. Giles, Oxford OX1 3LB, England, UK
e-mail: hitchin@maths.ox.ac.uk

H. Holden, R. Piene (eds.), *The Abel Prize*,
DOI 10.1007/978-3-642-01373-7_7, © Springer-Verlag Berlin Heidelberg 2010

Using the Hilbert space structure to define the adjoint of $A$, an alternative expression for the index is

$$\operatorname{ind} A = \dim \ker A - \dim \ker A^*.$$

As an example take $V = \ell^2$, the space of square-summable sequences $(a_0, a_1, a_2 \dots)$. If $A$ is the left shift

$$A(a_0, a_1, a_2 \dots) = (a_1, a_2, a_3 \dots)$$

its image is $V$ and kernel is one-dimensional spanned by $(1, 0, 0, \dots)$ so its index is 1, and the index of $A^n$ is $n$. For the right shift we get index $-1$ with powers of it giving all negative integers. An important property of the index is that a continuous deformation of $A$ through Fredholm operators leaves it unchanged. The dimension of the kernel may jump up and down, but the index is the same and determines the different connected components of the space of all Fredholm operators.

The Atiyah–Singer index theorem concerns itself with calculating this index in the case of an elliptic operator on a differentiable manifold. With suitable boundary conditions and function spaces these are Fredholm. The challenge, to which the theorem provides an answer, is to compute this integer in terms of topological invariants of the manifold and operator.

**2.2. Riemann–Roch.** In many respects the index theorem and its uses is modelled on the Riemann–Roch theorem for compact Riemann surfaces. Riemann was attempting to understand abelian integrals and meromorphic functions on a Riemann surface described by identification of sides of a polygon.

A meromorphic function $f$ on a Riemann surface is determined up to a constant multiple by its zeros $p_i$ and poles $q_j$ which are written as

$$(f) = \sum_i m_i p_i - \sum_j n_j q_j$$

where the integer coefficients are the multiplicities. An arbitrary expression like this—a finite set of points with multiplicities—is called a divisor. Not all of them come from a meromorphic function, but given a divisor $D$ one considers the dimension $\ell(D)$ of the vector space of meromorphic functions $f$ such that all the coefficients of $(f) + D$ are non-negative. Riemann established an inequality

$$\ell(D) \geq d + 1 - g$$

where $d$ is the *degree* of the divisor $D$—the integer $\sum_i m_i - \sum_j n_j$—and $g$ is the genus of the Riemann surface. These numbers are topological invariants, unchanged under continuous deformation. In particular, the Euler characteristic $\chi$ of the surface is $2 - 2g$. So the inequality estimates something analytical by topological means.

Riemann's inequality shows that there are many meromorphic functions on a Riemann surface and helped him to prove that any two were algebraically related which showed that many features of abstract Riemann surfaces could be reduced to algebraic plane curves.

Roch was a student of Riemann who died at the age of 26 in the same year 1866 that Riemann died. He identified the difference in Riemann's inequality in terms of a similar object to $\ell(D)$. What is now called the Riemann–Roch formula for curves is

$$\ell(D) - \ell(K - D) = d + 1 - g$$

where $K$ is the divisor of a meromorphic differential—it could be the derivative of a function or more generally an abelian differential. The left hand side is a difference of two positive integers, each one of which depends in general on the divisor $D$, but the right hand side is a topological invariant. This is an example of the index theorem but it needs a more modern interpretation to make it so.

There is a differential operator here—the Cauchy–Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y},$$

whose local solutions are holomorphic functions. The Riemann–Roch theorem is phrased above in terms of meromorphic functions—functions with singularities—but one gets around that by introducing the notion of a holomorphic line bundle associated to a divisor. So one considers complex-valued functions $f$ on the complement of the $p_i, q_j$ with specific behaviour near those points: near $p_i$ (with a local coordinate $z$ where $z = 0$ is $p_i$) the function $z^{-m_i} f(z, \bar{z})$ is differentiable and similarly at the points $q_j$, $z^{n_j} f(z, \bar{z})$ is differentiable. The space of all such functions is the infinite dimensional vector space of *sections* of a line bundle $L$. Because $z^{-m_i}$ and $z^{n_j}$ commute with the Cauchy-Riemann operator, there is a well-defined operator $\bar{\partial}$ on this space of sections. On suitable Sobolev spaces it defines a Fredholm operator, whose kernel has dimension $\ell(D)$. The dimension of its adjoint is $\ell(K - D)$, so the Riemann–Roch formula is

$$\dim \ker \bar{\partial} - \dim \operatorname{coker} \bar{\partial} = d + 1 - g.$$

Riemann's proof was heavily criticized because of its use of the physically inspired Dirichlet principle (not unlike some of the modern day incursions of physicists' thinking into pure mathematics) and a desire for more rigour propelled the theorem more into the algebraic domain after Riemann's death [22]. Nevertheless, its value was undeniable and indeed its use in the 19th century reflects many of the uses of the index theorem 100 years later: when $d$ is large enough, the right hand side is positive so the theorem asserts the existence of holomorphic sections of $L$, and if the degree of $K - D$ is less than zero, $\ell(K - D)$ vanishes and we get an exact formula. The theorem plus an additional vanishing theorem can be very powerful.

**2.3. The Beginning.** In 1961/62, Atiyah's first academic year in Oxford after moving from Cambridge, Singer decided to take a sabbatical from MIT. Remembering his friendship with Atiyah at the Institute for Advanced Study in Princeton in 1955, he had called to see if he could come on his own money and was of course welcome. Then, as Singer recalls, in January 1962 [28]:

> ... on my second day at the Maths Institute you walked up to the fourth floor office where I was warming myself by the electric heater. After the usual formalities, you asked "Why is the genus an integer for spin manifolds?" "What's up, Michael? You know the answer much better than I." "There's a deeper reason," you said.

And so began the Atiyah–Singer Index Theorem.

To understand the background to Atiyah's question, one has to understand the changes that had happened in geometry since Riemann's time. Riemann had invented the concept of a manifold, a higher dimensional version of a surface, but it took lifetimes for the idea to be properly understood. By 1962 however these were familiar objects and their structure was being analyzed from many different points of view. The rapid development of topology in the first half of the 20th century had provided a sophisticated algebraic setting for many of the invariants—much of it encoded in the cohomology ring. And de Rham's cohomology theory gave an analytical hold on this, representing cohomology classes by exterior differential forms. Then Hodge had showed that, with a Riemannian metric on the manifold, one could find a unique harmonic form in each cohomology class. When applied to algebraic surfaces it showed that holomorphic differentials were closed and provided a link to topology which had held up the further development of the Riemann–Roch theorem since the 19th century.

In the immediate postwar period the notion of a vector bundle—a family of vector spaces parametrized by the manifold—and in particular the tangent bundle, had come into play and the characteristic cohomology classes named after Pontryagin and Chern were the subject of great study. Most notably, Friedrich Hirzebruch, who had also been in Princeton in 1955 had come up with a means of describing the *signature* of a manifold in terms of particular combinations of Pontryagin classes.

**2.4. The Signature.** In the present context it is convenient to use de Rham cohomology to define the signature. The cohomology group $H^p(M,\mathbf{R})$ consists of the quotient space of the space of differential $p$-forms $\alpha$ such that $d\alpha = 0$ (*closed* forms) modulo those for which $\alpha = d\beta$ for some $(p-1)$-form $\beta$ (*exact* forms). Here a $p$-form is written in local coordinates as

$$\alpha = \sum_{i_1<i_2<\cdots<i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

and then

$$d\alpha = \sum_{j,i_1<i_2<\cdots<i_p} \frac{\partial a_{i_1 i_2 \ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

For a compact orientable manifold of dimension $n$, $H^n(M,\mathbf{R})$ is one-dimensional and the exterior product of forms defines a dual pairing between $H^p(M,\mathbf{R})$

and $H^{n-p}(M,\mathbf{R})$:

$$([\alpha],[\beta]) = \int_M \alpha \wedge \beta. \tag{2.1}$$

If we introduce a Riemannian metric $g_{ij}$ then there is a naturally defined volume form $\omega = \sqrt{\det g_{ij}}\,dx_1 \wedge \cdots \wedge dx_n$ and an inner product on forms. The *Hodge star operator* is the linear map $* : \Omega^p \to \Omega^{n-p}$ from the space of all $p$-forms to $(n-p)$-forms with the property that at each point

$$(\alpha,\beta)\omega = \alpha \wedge *\beta.$$

We have $*^2 = (-1)^{p(n-p)}$ when $*$ acts on $p$-forms.

The formal adjoint $d^*$ of $d$ satisfies the condition

$$\int_M (d\alpha,\beta)\omega = \int_M (\alpha, d^*\beta)\omega$$

and can be written using the star operator as

$$d^* = (-1)^{np+n+1} * d * . \tag{2.2}$$

The Hodge theorem says that in each cohomology class there is a unique representative form which satisfies $d\alpha = 0$ and $d^*\alpha = 0$.

Hodge theory immediately implies that the pairing (2.1) is non-degenerate since if $([\alpha],[\beta]) = 0$ for all $[\beta]$ then in particular we can take $\beta = *\alpha$ where $\alpha$ is harmonic (from (2.2) $\beta$ is closed). This implies that

$$0 = \int_M (\alpha, *^2\alpha) = \pm \int_M (\alpha,\alpha)$$

and so $\alpha = 0$.

If $n = 2m$ is even, we obtain a nondegenerate bilinear form on $H^m(M,\mathbf{R})$. This is symplectic when $m$ is odd (since odd forms anticommute) and symmetric when $m$ is even. In the latter case there is a basis in which the matrix is diagonal with $p$ positive entries and $q$ negative ones. The *signature* $\tau(M)$ of the manifold $M$ is defined to be the integer $p - q$.

The signature has some very natural properties: for a product $\tau(M \times N) = \tau(M)\tau(N)$ and for a change of orientation (replacing the volume form $\omega$ by $-\omega$) we clearly get $-\tau(M)$. Most importantly, if $M$ is the boundary of another oriented manifold of one dimension higher, then $\tau(M) = 0$.

Now in the mid 1950s René Thom had developed the theory of cobordism—considering equivalence classes of closed manifolds under the relation that two $n$-dimensional manifolds are equivalent if there is an $(n+1)$-dimensional manifold whose boundary has two components $M$ and $N$. One then introduces a ring structure on the equivalence classes using the two operations of product and disjoint union. Introducing orientations, one writes $[-M]$ for the class $M$ with opposite orientation and then $[M] + [-M] = 0$ by consideration of the cylinder $M \times [0,1]$.

By the remarks above, $\tau$ defines a homomorphism from the cobordism ring to the integers.

Over the rational numbers, Thom determined this ring: he showed that a class is determined by the Pontryagin numbers of the tangent bundle and also gave generators. The Pontryagin numbers play an important role in the index theorem, so let us look more closely at these.

The basic topological invariant of a *surface* is its Euler characteristic—for a triangulation it is $V - E + F$ where $V, E, F$ are the numbers of vertices, edges and faces respectively. It is also quite familiar that this number can be calculated by the number of zeros of a vector field, counted with sign and multiplicity. For every oriented vector bundle of rank two on a manifold $M$, there is a cohomology class in $H^2(M, \mathbf{Z})$ called the *Euler class* which when evaluated on a surface in $M$ counts the number of zeros of a section. In the case of the tangent bundle of a surface a section is a vector field and so this number is the Euler characteristic. For a Riemann surface, a holomorphic line bundle is a complex vector bundle of rank one which can be thought of as a real rank two bundle and this number is the degree which appears on the right hand side of the Riemann–Roch formula. The Euler class changes sign if we change the orientation of the bundle—evaluating it on a surface necessitates also a choice or orientation on the surface so the integer (for example the Euler characteristic itself) does not depend on orientation.

Now suppose a rank four bundle $E$ is a direct sum $E_1 \oplus E_2$ of two rank two bundles. We have two Euler classes $e_1, e_2 \in H^2(M, \mathbf{Z})$. The signs are indeterminate as is their order, but the class $e_1^2 + e_2^2 \in H^4(M, \mathbf{Z})$ is insensitive to this. If we have an overall orientation on $E$ then there is another class $e_1 e_2 \in H^4(M, \mathbf{Z})$ which is well-defined. The first example is called the (first) Pontryagin class of $E$. It makes sense even if $E$ is not a direct sum (by a trick called the splitting principle one can pass to another space over which the bundle does split as a sum without losing information; so most calculations can be performed by imagining that the bundle does split). For a vector bundle of rank $2m$ we define the Pontryagin class $p_k \in H^{4k}(M, \mathbf{Z})$ using the $k$th elementary symmetric function in $e_1^2, e_2^2, \ldots, e_k^2$. There is also, with an orientation, a class $e_1 e_2 \ldots e_k \in H^{2k}(M, \mathbf{Z})$ called the Euler class. A *Pontryagin number* of a compact manifold of dimension $4k$ is obtained by taking the Pontryagin classes of the tangent bundle and evaluating a degree $4k$ class

$$p_{i_1} p_{i_2} \cdots p_{i_n}$$

where $(i_1, i_2, \ldots, i_n)$ is a partition of $k$. The Pontryagin number is an integer.

Let $Q(x)$ be a power series with $Q(0) = 1$ and rational coefficients then the product

$$Q(e_1^2) Q(e_2^2) \ldots Q(e_k^2)$$

is a series whose terms are of degree $0, 4, 8, \ldots$ and each term of a given degree is a symmetric polynomial in the $e_i^2$ and hence a polynomial in Pontryagin classes. The degree $4k$ component can be evaluated on a manifold of dimension $4k$ to give a rational combination of Pontryagin numbers. This number $q(M)$ satisfies the condition that $q(M \times N) = q(M)q(N)$. Moreover since the Pontryagin classes themselves are independent of orientation, when we evaluate on the manifold we need a choice of orientation, so the numbers $q(M)$ change sign if we change the orientation. Thom's result that the cobordism ring is determined rationally by the Pontryagin numbers means that $q$ defines a ring homomorphism to $\mathbf{Q}$.

Hirzebruch's task was to find the function $Q$ for which this homomorphism is the signature, and he discovered that it was

$$Q(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}.$$

Expanding this in symmetric polynomials and substituting for the Pontryagin classes gives

$$L = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots$$

so Hirzebruch's theorem says that the signature of a $4k$-dimensional manifold is the Pontryagin number of degree $4k$ in this expansion. Because of the cobordism invariance, all one has to do is to check both sides on generators of the cobordism ring, which are standard well-known manifolds.

**2.5. Hirzebruch–Riemann–Roch.** Hirzebruch followed up his work on the signature with a version of the Riemann–Roch theorem for algebraic varieties of arbitrary dimension, not just Riemann's original one-dimensional case. This work appeared in the highly influential book of 1956 [24]. Whereas the original theorem related the dimensions of two vector spaces of holomorphic sections of a line bundle, the higher-dimensional case involves more complicated objects. These are most conveniently described by the Dolbeault approach.

On a complex manifold $M$ of complex dimension $n$, one can consider not only the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ but also an analogue on $(0, p)$ forms: a $(0, p)$-form is locally written in complex coordinates $z_i$ as

$$\alpha = \sum_{i_1 < i_2 < \cdots < i_p} a_{i_1 i_2 \ldots i_p}(x) d\bar{z}_{i_1} \wedge d\bar{z}_{i_2} \wedge \cdots \wedge d\bar{z}_{i_p}$$

and then

$$\bar{\partial}\alpha = \sum_{j, i_1 < i_2 < \cdots < i_p} \frac{\partial a_{i_1 i_2 \ldots i_p}}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_{i_1} \wedge d\bar{z}_{i_2} \wedge \cdots \wedge d\bar{z}_{i_p}.$$

By analogy with de Rham cohomology one defines the Dolbeault cohomology group $H^{0,p}$ as the kernel of $\bar{\partial}$ on $(0, p)$-forms modulo the image of $\bar{\partial}$ on $(0, p-1)$-forms. One can also incorporate a holomorphic vector bundle $E$ and consider the associated operator on forms with values in $E$. When $p = 0$, the kernel of $\bar{\partial}$ is simply the space of global holomorphic sections of $E$. Hirzebruch gave a formula for the alternating sum

$$\sum_{p=0}^{n}(-1)^p \dim H^{0,p}(M,E)$$

in terms of topological invariants which are the complex analogues of Pontryagin classes, called *Chern classes*.

A complex line bundle $L$ defines a class $c(L) \in H^2(M,\mathbf{Z})$—considered as a real oriented rank two bundle this is the Euler class $e$. If a complex vector bundle $E$ of rank $m$ splits as a sum of line bundles $L_i$, then the elementary symmetric functions in $c(L_i)$ define the Chern classes $c_k(E) \in H^{2k}(M,\mathbf{Z})$, and we can form Chern numbers instead of Pontryagin numbers by evaluating products in degree $2n$ on the manifold $M$, and use power series $Q(x)$. Hirzebruch developed a method closely related to his proof of the signature theorem which allowed him to find the right combination of Chern numbers to give the value of the alternating sum, by evaluating both sides on some standard examples. In the case without the vector bundle $E$ his formula is

$$\sum_{p=0}^{n}(-1)^p \dim H^{0,p}(M) = \operatorname{td}(TM)[M]$$

where td is the *Todd polynomial* defined by evaluating the Chern numbers of the tangent bundle generated by the polynomial

$$Q(x) = \frac{x}{1-e^{-x}}$$

which gives

$$\operatorname{td} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots .$$

With a vector bundle $E$, one introduces another polynomial in symmetric functions, the *Chern character*, defined by

$$\operatorname{ch}(E) = \sum_i e^{c(L_i)} = \operatorname{rk} E + c_1(E) + \frac{1}{2}(c_1^2 - 2c_2)(E) + \cdots$$

and then the general Riemann–Roch formula is

$$\sum_{p=0}^{n}(-1)^p \dim H^{0,p}(M,E) = \operatorname{ch}(E)\operatorname{td}(TM)[M].$$

When $M$ is one-dimensional, a Riemann surface, and $E$ is rank one, a line bundle $L$, the right hand side is

$$(1+c(L))\left(1+\frac{1}{2}c_1(TM)\right)[M] = \deg L + \frac{1}{2}(2-2g) = d+1-g$$

which is the right hand side of the classical Riemann–Roch theorem. The left hand side is

$$\dim H^{0,0}(M, L) - \dim H^{0,1}(M, L).$$

To link this with the traditional formulation one needs the Serre duality theorem which in general asserts that

$$H^{0,p}(M, E)^* \cong H^{0,n-p}(M, E^* \otimes K)$$

where $K$ is the canonical bundle of holomorphic $n$-forms. The struggles of the 19th century geometers to obtain a Riemann–Roch theorem for algebraic surfaces may well have been reflected by the inability to come to terms with higher cohomology—Serre duality does not convert the $H^{0,1}$ term into anything more amenable.

Hirzebruch showed that the Todd polynomial was closely related to Pontryagin classes. He introduced the $\hat{A}$ polynomials in Pontryagin classes defined by the power series

$$Q(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}$$

giving

$$\hat{A} = 1 - \frac{1}{24}p_1 + \frac{1}{2^7 3^2 5}(-4p_2 + 7p_1^2) + \cdots \tag{2.3}$$

and he showed that

$$\mathrm{td}(TM) = e^{c_1(TM)/2}\hat{A}(TM). \tag{2.4}$$

All of these formulae provoke an obvious question—the right hand side is a *rational* combination of Pontryagin numbers and so a priori doesn't give an integer, though the interpretation of the left hand side—either the signature or an alternating sum of dimensions—clearly is. Algebraic topologists were explaining this by quite sophisticated methods in the early 1960s, and the question that Atiyah asked Singer in January 1962 was motivated by one of these, relating precisely to the $\hat{A}$ polynomial above. On an algebraic variety with $c_1(TM) = 0$ the Hirzebruch–Riemann–Roch formula shows that $\hat{A}(TM)[M]$ is an integer. Hirzebruch had also shown that a weaker result holds. The mod 2 reduction of $c_1(TM) \in H^2(M, \mathbf{Z})$ is an invariant called the second Stiefel–Whitney class $w_2(TM) \in H^2(M, \mathbf{Z}_2)$ which exists on any manifold, complex or not. It was known that for any oriented manifold with $w_2 = 0$, the $\hat{A}$-genus was an integer. Why? As Atiyah commented: "We had the answer: we didn't know what the problem was" [2].

**2.6. The Dirac Operator.** By March 1962 Atiyah and Singer had found a candidate for the problem—determine the index of the Dirac operator. In a way they had rediscovered this operator since physicists were already familiar with it, but there was a huge difference between the Euclidean signature of Riemannian geometry which was needed here and the Lorentzian signature of relativity.

The construction of the Dirac operator begins with the Clifford algebra: the algebra generated by the vectors in a real vector space $V$ with positive definite inner product and the single relation

$$v^2 = -(v, v)1.$$

When $V = \mathbf{R}$ this gives the complex numbers, when $V = \mathbf{R}^2$ the quaternions. If $e_1, \dots, e_n$ is an orthonormal basis in $\mathbf{R}^n$ then the Dirac operator

$$D = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$$

has the property

$$D^2 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

What does $D$ act on? The complexified Clifford algebra is isomorphic to a matrix algebra in even dimensions $n = 2m$ and so $D$ acts on functions with values in this $2^m$ dimensional space of *spinors*.

On a Riemannian manifold each tangent space has an inner product and so one gets a bundle of Clifford algebras. But finding a global rank $2^m$ bundle $S$ on which this acts requires a topological constraint, satisfied if the second Stiefel–Whitney class $w_2(X) = 0$. This condition is therefore necessary for the existence of a global Dirac operator. A manifold satisfying this condition is called a *spin manifold*. If the manifold is not simply-connected there is a finite choice to be made of spin structures and Dirac operators.

Atiyah and Singer had been made aware of some of the results of Gelfand and his coworkers on the homotopy invariance of indices of elliptic boundary value problems [19, 20] and so a potential link with differential operators was already in the air. They conjectured that the $\hat{A}$ polynomial should give the index of a Dirac operator, to explain the integrality puzzle. The Dirac operator on its own is self-adjoint and so has zero index but the bundle $S$ can be broken up further according to the two half-spin representations. The volume form $\omega$ represents in the Clifford algebra an element such that $\omega^2 = (-1)^m$ and its two eigenspaces define a decomposition $S = S_+ \oplus S_-$ of the spinor bundle. For a vector $v \in V$, $v\omega = -\omega v$ in even dimensions, so the Dirac operator can be viewed as

$$D : C^\infty(S_+) \to C^\infty(S_-).$$

The index of this operator should be the $\hat{A}$-genus.

The other integrality questions were also amenable to an index interpretation. In fact the simplest is the Euler characteristic itself:

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p(M) = \dim H^{even} - \dim H^{odd}.$$

The operator

$$d + d^* \colon \Omega^{even} \to \Omega^{odd}$$

has by Hodge theory a kernel isomorphic to $H^{even}$ and a cokernel isomorphic to $H^{odd}$ so the index is the Euler characteristic.

Similarly

$$\sum_{p=0}^{n} (-1)^p \dim H^{0,p}(M, E) = \dim H^{0,even}(M, E) - \dim H^{0,odd}(M, E)$$

is the index of

$$\bar{\partial} + \bar{\partial}^* : \Omega^{0,even}(E) \to \Omega^{0,odd}(E)$$

and the Hirzebruch–Riemann–Roch theorem was explained as an index. Moreover, the theorem now held for an arbitrary complex manifold.

Finally Hirzebruch's signature theorem could be explained by considering, on a manifold of dimension $n = 2m$, the involution $\alpha : \Omega^p \to \Omega^{2m-p}$ defined by

$$\alpha = i^{p(p+1)} * .$$

This operator anticommutes with $d + d^*$ and, just as in the case of the Dirac operator, if we consider the $\pm$ eigenspaces we get an operator

$$d + d^* : \Omega_+ \to \Omega_-$$

and an index

$$\dim H_+(M) - \dim H_-(M).$$

When $p < m$, $\alpha$ preserves $H^p(M) \oplus H^{2m-p}(M)$ and identifying $H^p(M)$ and $H^{2m-p}(M)$ using $*$, $\alpha$ is

$$\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that the number of $+1$ and $-1$ eigenvalues is the same. Hence the index is the difference of these dimensions just for $p = m$, in the middle degree $H^m(M)$, which is the signature.

The integrality results could be explained by indices of operators, but the passage from the operator to the Pontryagin number required a theorem which became the index theorem. A proof was finally completed in the autumn of 1962 as Atiyah visited Harvard, and the results were presented in a seminar run by Bott and Singer, subsequently expounded in detail in the Princeton seminar [27]. The proof was based on Hirzebruch's 1953 proof of the signature theorem: the index theorem can be reduced to the evaluation of special cases which generate the cobordism classes. For the index problem one has to describe its change under cobordism and this required an extension of elliptic boundary value techniques to singular integral operators.

## 3 The Integer Index

**3.1. Formulation of the Theorem.** There are many variants of the Atiyah–Singer index theorem. We begin with the standard version, where the index is simply an integer. We start with a linear elliptic differential operator

$$D : C^{\infty}(V_+) \to C^{\infty}(V_-)$$

on a compact manifold $M$ with vector bundles $V_+,\ V_-$. Ellipticity is a property of the highest order term, the principal *symbol.*

A differential operator of order $r$ is locally expressible (using multi-index notation) as

$$Df = \sum_{|\alpha| \leq r} a_{\alpha} D^{\alpha} f$$

on vector-valued functions $f$. This means we have locally trivialized the bundle to write a section as the function $f$. In another trivialization $f$ is changed to $Pf$ for some invertible matrix-valued function $P$. The coefficients $a_{\alpha}$ of order less that $r$ transform involving a derivative of $P$ but the highest order terms do not. The symbol is invariantly defined as a section of

$$S^r T \otimes \mathrm{Hom}(V_+, V_-)$$

where $S^r T$ is the bundle of symmetric rank $r$ tensors. We can also think of it as a section $\sigma$ of $p^* \mathrm{Hom}(V_+, V_-)$ on the cotangent bundle $p : T^*M \to M$, homogeneous of degree $r$ along the fibres. The operator is *elliptic* if $\sigma(\xi)$ is invertible for $\xi \neq 0$. For example, the Dirac operator is elliptic because its symbol is the Clifford product $\sigma(\xi)\psi = \xi \cdot \psi$ and

$$\sigma(\xi)^2 \cdot \psi = -(\xi, \xi)\psi$$

where $(\xi, \xi) \geq 0$ is the Riemannian inner product on one-forms.

Ellipticity depends only on the principal symbol as does the index, which also is homotopy invariant. To obtain a topological object from it, we use the fact that it gives an isomorphism $V_+ \cong V_-$ outside the zero-section of $T^*M$ and defines a class in the cohomology with compact supports $H_c^*(T^*M)$.

We explain further: if $X$ is a non-compact manifold, we can consider the de Rham cohomology of differential forms with compact supports. So, for example although $H^n(\mathbf{R}^n) = 0$ for $n > 0$, $H_c^n(\mathbf{R}^n) \cong \mathbf{R}$, represented by $\varphi dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ where $\varphi \geq 0$ vanishes outside a compact set.

A *connection* $A$ on a vector bundle $V$ is a first order differential operator $d_A : C^{\infty}(V) \to C^{\infty}(V \otimes T^*)$ whose symbol is $\sigma(\xi)s = s \otimes \xi$. It extends to an "exterior derivative" operator on $p$-forms with values in $V$:

$$d_A : \Omega^p(V) \to \Omega^{p+1}(V)$$

but $d_A^2$ is no longer zero, and instead defines a *curvature* form $F_A \in \Omega^2(\mathrm{End}(V))$. We meet here the Chern–Weil theory, descendant of the classical Gauss–Bonnet theorem. In de Rham cohomology, the Pontryagin and Chern classes are represented by closed differential forms obtained by evaluating certain polynomials of matrices on the curvature form.

In our case we take the Chern character of $E$, which is represented in de Rham cohomology by the closed form

$$\sum_k \frac{1}{(2\pi i)^k} \operatorname{tr}(F_A^k).$$

In the situation above we choose connections $A, B$ on $V_+$ and $V_-$ and pull them back to $T^*M$. Outside some neighbourhood of the zero section of $T^*M$, $\sigma^* B = A + a$ where $a \in \Omega^1(\operatorname{End} V)$. Extending $a$ to the whole of $T^*M$ we have connections $A_+, A_-$ on $p^*V_+, p^*V_-$ which are equivalent by the isomorphism $\sigma$ outside of a compact set, hence $\operatorname{ch}(p^*V_+) - \operatorname{ch}(p^*V_-)$ defines a compactly supported cohomology class in $H_c^*(T^*M)$. This is the topological data derived from the operator $D$.

Now the cotangent bundle $T^*M$ is naturally a symplectic manifold. Using a Riemannian metric on $M$ its tangent bundle becomes a complex vector bundle and we can take its Todd class $\operatorname{td}(T)$. Then the Atiyah–Singer index theorem can be formulated as:

**Theorem 3.1** *Let $D : C^\infty(V_+) \to C^\infty(V_-)$ be an elliptic operator on a compact manifold. Then*

$$\operatorname{ind} D = \dim\ker D - \dim\operatorname{coker} D = \int_{T^*M} (\operatorname{ch}(V_+) - \operatorname{ch}(V_-))\operatorname{td}(T).$$

In many applications it is the Dirac operator coupled to a vector bundle which is the relevant operator. For the Dirac operator alone the formula is

$$\operatorname{ind} D = \hat{A}(TM)[M].$$

If $E$ is an auxiliary bundle with connection $d_A$ we define the Dirac operator with coefficient bundle $E$ as the composition of

$$\nabla \otimes 1 + 1 \otimes d_A : C^\infty(S_+ \otimes E) \to C^\infty(S_+ \otimes E \otimes T^*)$$

(where $\nabla$ is the Levi-Civita connection) with the Clifford multiplication map

$$S_+ \otimes E \otimes T^* \to S_- \otimes E$$

defined by $\varphi \otimes \xi \otimes e \mapsto \xi\varphi \otimes e$. The index formula for this operator is

$$\operatorname{ind} D_E = \operatorname{ch}(E)\hat{A}(TM)[M].$$

The example of the elliptic operator

$$d + d^* : \Omega^{even} \to \Omega^{odd}$$

arising from the de Rham complex

$$\cdots \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \cdots$$

gives rise to the associated idea of an *elliptic complex*

$$\cdots C^\infty(V^{p-1}) \xrightarrow{D_{p-1}} C^\infty(V^p) \xrightarrow{D_p} C^\infty(V^{p+1}) \xrightarrow{D_{p+1}} \cdots$$

where ellipticity means that if $\xi \neq 0$ and $\sigma_p$ is the symbol of $D_p$ then $\sigma_p(\xi)v = 0$ implies that $v = \sigma_{p-1}(\xi)w$. By choosing inner products on the $V^p$, the elliptic complex generates an elliptic operator

$$D + D^* \colon C^\infty(V^{even}) \to C^\infty(V^{odd})$$

just as in the de Rham complex and the index is the alternating sum of the dimensions of the cohomology spaces.

**3.2. Integrality Theorems.** The index theorem provided an explanation for many of the previously known and slightly puzzling integrality theorems, especially when the central role of the Dirac operator was appreciated. In the first place, there was the link with Riemann–Roch. If the vector space $V$ has a Hermitian inner product then the space $\oplus_0^m \Lambda^{0,p} V$ is a module over the Clifford algebra: given $v \in V$, take its (0, 1) part $v^{0,1}$ and define

$$v\varphi = \frac{1}{\sqrt{2}}(e(v) - e(v)^*)\varphi$$

where $e(v)$ is the exterior product by $v^{0,1}$ and $e(v)^*$ its adjoint. Taking $V$ to be the tangent space to a complex manifold with a Hermitian metric, this shows that the symbol of the Dirac operator on the bundle $\oplus_0^m \Lambda^{0,p} T^*$ is essentially the same as the symbol of $\bar\partial + \bar\partial^*$ and so they have the same index.

But then we don't need complex coordinates to get integrality of the Todd genus $\mathrm{td}(TM)[M]$ because it is just the index of a Dirac operator. This means (as was known) that the Todd genus of an *almost complex* manifold is an integer.

The space of (0, $p$) forms considered as a Clifford module is not the standard one—for that we need to choose a square root $L$ of the canonical bundle, a line bundle $L$ such that $L^2 \cong K$. The topological obstruction to finding that is the condition

$$c_1(T) \bmod 2 = c_1(K) \bmod 2 = w_2 = 0$$

and then the standard Dirac operator is equivalent to the $\bar\partial + \bar\partial^*$ operator with values in the line bundle $L = K^{1/2}$. This explains Hirzebruch's link between the Todd polynomials and the $\hat{A}$ polynomials (2.4).

The Dirac operator when the coefficient bundle is the spin bundle itself is the $d + d^*$ bundle on exterior forms. In this case the signature theorem and the formula for the Euler characteristic can both be seen to be examples of index theorems for Dirac operators.

A particularly nice example of integrality is a proof of Rochlin's 1952 result that if a compact 4-manifold is oriented and has $w_2 = 0$, then its signature is divisible by 16. The index theory proof goes as follows: since $w_2 = 0$ the manifold has a Dirac operator whose index from (2.3) is

$$\hat{A} = \left(1 - \frac{1}{24}p_1 + \cdots\right)[M] = -\frac{1}{24}p_1(TM)[M].$$

But in four dimensions we have the isomorphism $Spin(4) \cong Sp(1) \times Sp(1)$ where $Sp(1)$ is the group of unit quaternions. This means that the Dirac operator is quaternionic hence its kernel and the kernel of its adjoint are quaternionic vector spaces. In particular as complex vector spaces they are even-dimensional. It follows that $p_1(TM)[M]/24$ is an *even* integer. On the other hand the signature is

$$\tau(M) = \left(1 + \frac{1}{3}p_1 + \cdots\right)[M] = \frac{1}{3}p_1(TM)[M]$$

and so is divisible by $48/3 = 16$.

**3.3. Positive Scalar Curvature.** We noted in Sect. 2.6 that in $\mathbf{R}^n$ if

$$D = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$$

then

$$D^2 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

so that the Dirac operator is a sort of square root of the Laplacian. On a curved Riemannian manifold, the corresponding formula involves an extra zero-order term:

$$D^2 = \nabla^*\nabla + \frac{1}{4}R$$

where $R$ is the *scalar curvature* of the metric and $\nabla^*$ is the formal adjoint of the covariant derivative $\nabla : C^\infty(S) \to C^\infty(S \otimes T^*)$. The operator $\nabla^*\nabla$ is non-negative.

This formula, originally due to Schrödinger in 1932 in the Lorentzian setting, was introduced in the Riemannian case by Lichnerowicz in 1963 [26] as an early application of the index theorem. If $D\varphi = 0$ then taking global inner products

$$0 = (D^2\varphi, \varphi) = (\nabla^*\nabla\varphi, \varphi) + \frac{1}{4}(R\varphi, \varphi).$$

Thus if $R > 0$, since $\nabla^*\nabla$ is non-negative, we must have $\varphi = 0$, whether $\varphi$ is a section of $S_+$ or $S_-$. It follows that the index of the Dirac operator $\hat{A}(TM)[M] = 0$.

The simplest example of this is the four dimensional K3 surface (for example the quartic $z_0^4+z_1^4+z_2^4+z_3^4=0$ in $\mathbf{C}P^3$) which has $w_2=0$ and signature $-16$ hence $\hat{A}(TM)[M]=2$. This manifold cannot admit a metric of positive scalar curvature. In fact Yau's proof of the Calabi conjecture showed that it does admit a metric of zero scalar curvature.

**3.4. Gauge-Theoretic Moduli Spaces.** One of the most useful applications of the index theorem is to the calculation of the dimension of certain moduli spaces of solutions to nonlinear equations. These include the anti-self-dual Yang–Mills equations on a four-manifold, the Seiberg–Witten equations, equations for Higgs bundles, magnetic monopoles, pseudo-holomorphic curves and others. These are all nonlinear partial differential equations whose linearization can be made elliptic. The index theorem then produces the expected dimension of the space of solutions and with a little further information Banach space implicit function theorems give manifold structures on the moduli space. As an example we shall take the anti-self-duality equations on a compact 4-manifold $M$ (see for example [17]).

Let $E$ be a complex vector bundle of rank $r$ over $M$ with a Hermitian metric. A connection $A$ on $E$ has a curvature form $F_A\in\Omega^2(\operatorname{End}E)$. The connection is called *anti-self-dual* if

$$*F_A=-F_A.$$

If $A$ preserves the Hermitian metric, then $F_A$ is skew-adjoint.

A gauge transformation $g$ is a unitary automorphism of $E$ and $g$ acts on the curvature by conjugation hence preserves the notion of anti-self-duality. We want to understand the moduli space—the space of all such connections modulo gauge equivalence. Elliptic operators appear when we look at the linearization of the problem. The derivative at $A$ of a one-parameter family of connections is given by $\dot{A}\in\Omega^1(\operatorname{End}E)$. If this arises from a one-parameter family of gauge equivalent connections then $\dot{A}=d_A\psi$ where $\psi\in\Omega^0(\operatorname{End}E)$. The derivative of the curvature form at $A$ is $d_A\dot{A}\in\Omega^2(\operatorname{End}E)$ so if this arises from a one-parameter family of *anti-self-dual* connections then $*d_A\dot{A}=-d_A\dot{A}$, or

$$d_A^+\dot{A}=0\in\Omega^2_+(\operatorname{End}E)$$

where the $+$ subscript indicates orthogonal projection onto the $+1$ eigenspace of $*$ on $\Omega^2(\operatorname{End}E)$.

We thus have a sequence of first order operators

$$\Omega^0(\operatorname{End}E)\stackrel{d_A}{\to}\Omega^1(\operatorname{End}E)\stackrel{d_A^+}{\to}\Omega^2_+(\operatorname{End}E).$$

Moreover since $d_A^2=F_A$ and $*F_A=-F_A$, it follows that $d_A^+d_A=0$ so this is a complex.

The linearization of our problem (the tangent space of the moduli space) is thus the kernel of $d_A^+$ (the infinitesimal deformations of anti-self-dual connections) modulo the image of $d_A$ (the deformations arising from gauge transformations). Harmonic theory for this complex tells us that this is isomorphic to the kernel of

$$d_A^* + d_A^+ : \Omega^1(\mathrm{End}\, E) \to \Omega^0(\mathrm{End}\, E) \oplus \Omega^2_+(\mathrm{End}\, E)$$

and this is an elliptic operator.

We now calculate the index of this. This is a practical example which demonstrates how indices can be computed without going into the proof of the theorem. Firstly note that it is in fact a Dirac operator, with coefficient bundle $S_+ \otimes \mathrm{End}\, E$:

$$D\colon C^\infty(S_- \otimes S_+ \otimes \mathrm{End}\, E) \to C^\infty(S_+ \otimes S_+ \otimes \mathrm{End}\, E)$$

so its index is

$$\begin{aligned}
&-\mathrm{ch}(S_+ \otimes \mathrm{End}\, E)\hat{A}(TM)[M] \\
&\quad = -\big(r^2 + \mathrm{ch}_2(\mathrm{End}\, E) + \cdots\big)\big(2 + \mathrm{ch}_2(S_+) + \cdots\big)\left(1 - \frac{1}{24}p_1 + \cdots\right) \\
&\quad = -2\,\mathrm{ch}_2(\mathrm{End}\, E) - r^2\left(-\frac{1}{12}p_1 + ch_2(S_+)\right).
\end{aligned}$$

The last term is $r^2$ times the index of

$$d^* + d^+\colon \Omega^1 \to \Omega^0 \oplus \Omega^2_+$$

which by Hodge theory is

$$b_1 - 1 - b_2^+ = \frac{1}{2}(2b_1 - 2 - b_2^+ - b_2^- - b_2^+ + b_2^-) = \frac{1}{2}\big(-\chi(M) - \tau(M)\big).$$

To calculate the first term note that

$$\begin{aligned}
&\mathrm{ch}(\mathrm{End}\, E) = \mathrm{ch}(E^* \otimes E) = \mathrm{ch}(E^*)\,\mathrm{ch}(E) \\
&\qquad = \left(r - c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots\right)\left(r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots\right)
\end{aligned}$$

so that

$$\mathrm{ch}_2(\mathrm{End}\, E) = \big[r(c_1^2 - 2c_2) - c_1^2\big](E)[M].$$

The final index is

$$-2\big[r(c_1^2 - 2c_2) - c_1^2\big](E)[M] - \frac{1}{2}r^2(\chi + \tau).$$

In the case of $M = S^4$, we have $b_2 = 0$, so $c_1(E) = 0$ and $\tau = 0$, $\chi = 2$ and the formula becomes $4rc_2(E) - r^2$.

This is just the index, but, as shown by Freed and Uhlenbeck, a deformation of the metric will make $d_A^* + d_A^+$ surjective. The kernel of $d_A$ is the space of covariant constant infinitesimal gauge transformations which for an irreducible connection is just the scalars, so the final dimension of the moduli space is the index plus one. In the case $r = 2$ on $S^4$ this gives the $8k - 3$ of [12]. The global study of the moduli space on a general four-manifold is of course the content of Donaldson theory.

The study of instantons on $S^4$ began with this index theoretical approach [12]. The subsequent ADHM description in terms of matrices also uses the index theorem [13], in this case for the Dirac operator

$$D : C^\infty(S_- \otimes E) \to C^\infty(S_+ \otimes E)$$

where the index is $-\operatorname{ch}(E)\hat{A}(TM)(M) = c_2(E) = k$. In this case the Lichnerowicz formula for $S_+ \otimes E$ is still just the scalar curvature term $R/4$ (positive for $S^4$) because the anti-self-dual curvature $F_A$ acts trivially on the spinors in $S_+$. Hence the index gives the actual dimension of the kernel of $D$. By stereographic projection these become $\mathcal{L}^2$ sections on $\mathbf{R}^4$ and the $k \times k$ ADHM matrices are the global inner products

$$(x_i \varphi_\alpha, \varphi_\beta)$$

for an orthonormal basis $\varphi_1, \ldots, \varphi_k$ of solutions to $D\varphi = 0$.

## 4 The Equivariant Index

**4.1. K-Theory.** The first proof of the index theorem was not flexible enough to support the myriad applications which Atiyah and Singer had in mind—in particular to study group actions and families. Hand in hand with the development of the index theorem came the development of K-theory—a generalized cohomology theory which was naturally adapted to considering families of vector spaces and not just the integer which is their dimension. A series of papers [3–7] in *Annals of Mathematics* in the period 1968–71 became the definitive version of the index theorem and K-theory was the basic tool this time. The model for the new proof was not Hirzebruch's use of cobordism, but Grothendieck's version of the Riemann–Roch theorem, replacing the algebraic K-theory groups defined in terms of coherent sheaves by the topological theory developed by Atiyah and Hirzebruch.

In particular, suppose one has a group $G$ acting on the manifold together with an action on vector bundles $V_+$ and $V_-$, preserving an elliptic operator

$$D : C^\infty(V_+) \to C^\infty(V_-).$$

Then the kernel and cokernel of $D$ are representation spaces of $G$. The plain integer index of $D$ simply gives the differences of the degrees of these representations and no further information, whereas the natural object to consider is a formal difference of representation spaces. This lies in the *Grothendieck group* $R(G)$ generated by representations of $G$—two formal differences $U - V$, $U' - V'$ define the same element in $R(G)$ if there is a representation $W$ such that $U \oplus V' \oplus W$ is isomorphic to $U' \oplus V \oplus W$.

The same construction applied to the natural numbers gives the ring of integers, and applied to isomorphism classes of vector bundles on a space $X$ (using the direct sum operation of vector bundles) defines $K(X)$, topological K-theory. This is also a ring under the operation of tensor product. If $X$ is a point, then a vector bundle is a vector space and then $U - V \mapsto \dim U - \dim V$ gives an isomorphism from $K(pt.)$ to the integers.

In general, since $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$ and $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$ the Chern character defines a homomorphism from $K(X)$ to the cohomology ring of $X$ but only with rational coefficients because of the denominators in

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

In pre-index theory days, integrality issues were being addressed by using K-theory, so it is not surprising that it became the natural setting for much of the development of the theory.

On a non-compact space $X$ one can define K-theory with compact supports $K_c(X)$ in terms of vector bundles together with an isomorphism outside a compact set. Then an open inclusion $i : U \subset X$ induces a natural map

$$i_! : K_c(U) \to K_c(X).$$

Clearly the symbol $\sigma$ of an elliptic operator defines an isomorphism between $p^* V_+$ and $p^* V_-$ over $T^* M$ outside the zero section so we immediately obtain a symbol class

$$[\sigma] \in K_c(T^* M).$$

For the usual index theorem we need to extract an integer out of this, and to do this one uses a consequence of the Bott periodicity theorem concerning the homotopy groups of unitary groups $U(n)$, for $n$ large. This has a rather different manifestation in K-theory—if $V$ is a complex vector bundle over a space $X$, and $v \in V$ then the exterior product $e(v)$ defines a complex

$$\cdots \xrightarrow{e(v)} \Lambda^p V \xrightarrow{e(v)} \Lambda^{p+1} V \xrightarrow{e(v)} \cdots$$

exact outside the zero section $v = 0$. This gives a class $\lambda_V \in K_c(V)$ and by tensoring with vector bundles pulled back from $X$ it defines a homomorphism

$$\varphi \colon K(X) \to K_c(V).$$

This is an isomorphism, called the Thom isomorphism.

To use this to define a topological invariant from the symbol class $[\sigma]$ one embeds the manifold $M$ in a large Euclidean space $\mathbf{R}^m$. If $N$ is the normal bundle of $M$ in $\mathbf{R}^m$ then $N$ can be identified with a tubular neighbourhood of $M$ in $\mathbf{R}^m$—an open subset—and hence we get an open embedding of $TN$ in $T\mathbf{R}^m$. Using the induced metric, we identify $T^* M$ and the tangent bundle $TM$ and define a complex structure on the tangent bundle of $TM$, of $T\mathbf{R}^m$ and hence the normal bundle of $TM$ in $T\mathbf{R}^m$. So we have the Thom isomorphism

$$\varphi\colon K_c(TM) \to K_c(TN),$$

the open inclusion $i : TN \subset T\mathbf{R}^m$ giving

$$i_!\colon K_c(TN) \to K_c(T\mathbf{R}^m),$$

and the Thom isomorphism theorem again giving

$$\varphi\colon K(pt.) \to K_c(\mathbf{C}^m) = K_c(T\mathbf{R}^m).$$

Using all these maps together with $TM \cong T^*M$ we get a homomorphism, called the *topological index*

$$\text{t}{-}\text{ind}\colon K_c(T^*M) \to \mathbf{Z}.$$

In this formulation, the integer index theorem says that the topological index of the symbol class $[\sigma] \in K_c(T^*M)$ is the analytical index of the elliptic operator $D$.

The above set-up is perfectly adapted to deal with the equivariant case, replacing the embedding $M \subset \mathbf{R}^m$ by an equivariant embedding in a representation space, possible by the Peter–Weyl theorem. The result then is that the symbol class lies in the equivariant K-theory $K_G(T^*M)$ and the topological index is in $K_G(pt.)$ which is the Grothendieck group for the representations of $G$.

**4.2. Fixed Point Theorems.** One of the remarkable features of the equivariant index is the ability to calculate it from data at the fixed point set of elements of the group $G$. The classical example of this is the Lefschetz fixed point theorem of 1926. Here, given a map $f\colon X \to X$, the Lefschetz number

$$\sum (-1)^p \operatorname{tr}(f_*|H^p(X))$$

is calculated in terms of the sum of certain indices at the fixed points. In good situations one gets a count of the number of fixed points.

Clearly if $G$ is a group of isometries of a Riemannian metric, it acts naturally on $p$-forms and commutes with the operator $d + d^*$. The alternating sum in the Lefschetz formula for $g \in G$ is then the *character* at $g$ of the equivariant index of

$$d + d^*\colon \Omega^{even} \to \Omega^{odd}.$$

However, there are more possibilities by taking different elliptic operators or complexes—the Dolbeault complex or the Dirac operator for example. Many of the consequences of this localization were spelled out in the papers [8, 9] of Atiyah and Bott. We mention here two applications. Both involve the action on the elliptic complex

$$\cdots \to \Omega^{0,p}(E) \xrightarrow{\bar\partial} \Omega^{0,p+1}(E) \xrightarrow{\bar\partial} \cdots$$

for a holomorphic vector bundle on which the group $G$ acts. In this case the concrete Lefschetz formula says that the alternating sum of the characters at $g$ of the action on $H^{0,p}(M, E)$ is, if $g$ has isolated fixed points,

$$\sum_{g(x)=x} \frac{\operatorname{tr} \varphi_x}{\det(1 - dg_x)} \tag{4.1}$$

where $\varphi_x \colon E_x \to E_x$ is the action on $E$ at the fixed point and $dg_x$ the action on the holomorphic tangent space at $x$.

For the first example we take a compact simply-connected simple Lie group $G$ and its maximal torus $T$. The flag manifold $G/T$ can be given the structure of a homogeneous Kähler manifold, and a homomorphism $T \to S^1$ given by the weight $\lambda$ defines a circle bundle whose associated homogeneous line bundle $L$ over $G/T$ is holomorphic. With appropriate choices the cohomology spaces $H^{0,p}(G/T, L)$ vanish for $p > 0$ and for $p = 0$ one gets an irreducible representation of $G$. This is the content of the Borel–Weil theorem—a direct realization of a representation given the maximal weight which is the initial character.

Because of the vanishing of the higher cohomology, the equivariant index for the $\bar\partial + \bar\partial^*$ operator is precisely the representation on $H^{0,0}(G/T, L)$. The index theorem gives a formula for its character.

To do this choose $g \in T$ such that the closure of the group it generates is $T$ itself. If $ghT = hT$, then $h^{-1}gh \in T$ and so taking powers and closure, $h^{-1}Th = T$. The fixed points are therefore in one-to-one correspondence with $N(T)/T$ where $N(T)$ is the normalizer of $T$, and this quotient is the Weyl group $W$, a finite group. The index theorem thus gives an expression for the character in terms of a sum over the Weyl group.

The tangent space at a point can be identified with $\mathfrak{g}/\mathfrak{t}$ which is a sum of 2-dimensional root spaces defined by the positive roots $\alpha_1, \ldots, \alpha_k$, so the denominator in the formula (4.1) at $w \in W$ is a product

$$\prod_1^k w(1 - e^{\alpha_i}).$$

The numerator is the action on the line bundle $L$ which is $w(e^\lambda)$ where $\lambda$ is the maximal weight. Since $G$ is simply connected, half the sum of the positive roots $\rho$ is a weight and $w(e^\rho) = \pm e^\rho = \operatorname{sgn}(w) e^\rho$, so we get the more familiar form for the Weyl character formula:

$$\frac{1}{e^\rho \prod_1^k (1 - e^{\alpha_i})} \sum_{w \in W} \operatorname{sgn}(w) w(e^{\lambda+\rho}).$$

A wide range of other examples can be obtained by considering the action of finite groups on algebraic surfaces. The index theorem then operates as a machine for producing identities in number theory. The article [25] gives a good survey of this. Here we take the signature operator and the holomorphic action of $g \in G$, a finite group, on an algebraic surface $M$. The fixed points are either isolated points $x \in M$ or algebraic curves $Y$. In the first case $g$ acts on the tangent space as $(e^{i\alpha} \oplus e^{i\beta})$ and in the second on the normal bundle by $e^{i\theta}$. The fixed point contribution to the equivariant index is then

$$-\cot\frac{\alpha}{2}\cot\frac{\beta}{2} \quad \text{or} \quad Y \cdot Y \operatorname{cosec}^2\frac{\theta}{2}$$

where $Y \cdot Y$ is the self-intersection number of the curve $Y$ (the degree of the normal bundle).

This is the fixed point contribution which the index theorem for the signature operator relates to the action of $G$ on $H^2(M, \mathbf{R})$. A simple example is the case where $M \subset \mathbf{C}P^3$ is the algebraic surface with equation in homogeneous coordinates

$$z_0^n + z_1^n + z_2^n + z_3^n = 0$$

and $g$ is the action of the $n$th root of unity $\omega$:

$$g(z_0, z_1, \ldots, z_3) = (\omega^{-1} z_0, z_1, \ldots, z_3).$$

The fixed point set of $g^k$ for $k \neq 0$ is the plane section $Y$ with equation $z_0 = 0$, which has self-intersection $n$. The equivariant index theorem then gives

$$\chi(g^k) = n \operatorname{cosec}^2 \frac{\pi k}{n}.$$

Now averaging the character over the group gives the degree of the trivial representation, which is

$$\frac{1}{n}\left(\sum_{k=1}^{n-1} \chi(g^k) + \tau(M)\right).$$

On the other hand, the invariant part of $H^2(M, \mathbf{R})$ can be interpreted as the cohomology of the quotient $M/G$ and the index is the signature of the invariant quadratic form on this. In our case, the quotient is $\mathbf{C}P^2$ which has signature 1. So we get

$$1 = \frac{1}{n}\left(\sum_{k=1}^{n-1} \chi(g^k) + \tau(M)\right) = \sum_{k=1}^{n-1} \operatorname{cosec}^2 \frac{\pi k}{n} + \frac{1}{n}\tau(M).$$

This offers two interpretations—the topologist would calculate the Chern classes of the surface $M$ as $c_1 = (4-n)h, c_2 = (6-4n+n^2)h^2$ and use the signature theorem to give $\tau(M) = (c_1^2 - 2c_2)/3 = n(4-n^2)/3$. Then the equivariant index theorem gives the number theoretic identity:

$$\sum_{k=1}^{n-1} \operatorname{cosec}^2 \frac{\pi k}{n} = \frac{n^2-1}{3}.$$

The number theorist would give an elementary proof of this and derive the signature of $M$.

This is a simple example, but when isolated points occur as fixed points, the contributions there have appeared in the classical literature as *Dedekind symbols* and results of Rademacher and Mordell can be obtained this way as well as many more identities.

**4.3. Rigidity Theorems.** The equivariant index theorem together with its fixed-point formulation enables a character to be evaluated in two different ways. As above, this provides a route to identities which can also be proved by other means with the appropriate skills. The same idea, however, also leads to some remarkable results about the degree of symmetry, or rather lack of it, of certain manifolds. For example, Atiyah and Hirzebruch showed in [10] that a manifold $M^{4k}$ with $w_2 = 0$ and $\hat{A}[M] \neq 0$ admits no non-trivial circle action. Recall from Sect. 3.3 that the same hypotheses prohibit the existence of a metric of positive scalar curvature, so this result is identifying a range of manifolds at the opposite extreme from those with positive curvature and homogeneous, like spheres. The method has since been radically extended through ideas of Witten [16].

An $S^1$-invariant elliptic operator $D$ is called *rigid* if the equivariant index is trivial as a representation, in other words if the non-trivial representations occur with the same multiplicity in the kernel and cokernel of $D$. For the $d + d^*$ operator or the signature operator this is clear since a diffeomorphism in the circle action is connected to the identity and so acts trivially on cohomology by homotopy invariance. The Hodge theorem then implies it acts trivially on the kernel of $d + d^*$. It also becomes transparent when using the equivariant index formula for isolated fixed points. For

$$d + d^* : \Omega^{even} \to \Omega^{odd}$$

the fixed point contribution is just $+1$ (this is the original Lefschetz fixed point formula). For the signature operator on $M^{4k}$ the contribution is

$$\prod_1^{2k} \frac{1 + e^{im_j\theta}}{1 - e^{im_j\theta}}$$

where the tangent space breaks up into 2-dimensional pieces on which the circle acts as $e^{im_j\theta}$.

The equivariant index theorem says that the character for a generic $g$ in the circle is the sum over fixed points

$$\sum_{g(x)=x} \prod_1^{2k} \frac{1 + e^{im_j\theta}}{1 - e^{im_j\theta}}.$$

But the character is a finite sum of terms of the form $e^{im\theta}$ so we get an identity of meromorphic functions

$$\sum_{i=-N}^{N} a_i z^i = \sum_{g(x)=x} \prod_1^{2k} \frac{1+z^{m_j}}{1-z^{m_j}}.$$

But the left hand side is a finite Laurent series and so has poles only at $z=0$ whereas the right hand side has poles on the unit circle. Both sides must therefore equal a constant function.

When $w_2 = 0$ we have a Dirac operator and here the contribution is

$$\prod_1^{2k} \frac{z^{m_j/2}}{1-z^{m_j}}$$

(the factor $1/2$ involves a slightly subtle lifting of the circle action to the spin structure). In this case the right hand side vanishes when $z=0$ which shows that the equivariant index is not just constant but is zero, which is the theorem of Atiyah and Hirzebruch in this case. The general proof involves the consideration of fixed point sets of arbitrary dimension.

Witten's extension of this (given mathematical proof in [16]) introduces Dirac operators whose coefficient bundles are derived from the tangent bundle in a specific manner. If $S^kT$ and $\Lambda^kT$ denote the symmetric and exterior powers of the tangent bundle, one writes

$$S_t = \sum_0^\infty t^k S^k T \qquad \Lambda_t = \sum_0^\infty t^k \Lambda^k T$$

and

$$R_q = \sum_0^\infty q^n R_n = \bigotimes_{n=1}^\infty \Lambda_{q^n} \bigotimes_{m=1}^\infty S_{q^m}$$

and

$$R'_q = \sum_0^\infty q^{n/2} R'_n = \bigotimes_{n=1}^\infty \Lambda_{q^{(2n+1)/2}} \bigotimes_{m=1}^\infty S_{q^m}$$

The theorem is then that the Dirac operator with coefficient bundle $R'_n$ and the signature operator with bundle $R_n$ are rigid.

While the mathematical proof of this is a consequence of the equivariant index theorem, understanding the reasons for this rigidity is more demanding than finding alternative proofs for number-theoretical identities. In Witten's derivation using the loop space of $M$, the modular property of the polynomial $Q(x)$ which generates these particular genera plays a fundamental role which has not yet been fully absorbed into the mathematics.

## 5 The mod 2 Index

**5.1. Real K-Theory.** There was one development of index theory which, in Atiyah's words, "could easily have been missed out at the first step ... and trodden underfoot in the stampede later on" [2]. This involves a mod 2 invariant which is not definable in terms of cohomology classes. In fact there are manifolds homotopically equivalent to spheres for which it is non-zero.

It is K-theory which reveals its presence. We mentioned in Sect. 4.1 that Bott periodicity lay behind the isomorphism $K_c(\mathbf{C}^n) \cong \mathbf{Z}$. The point is that a vector bundle which is trivial outside a compact set in $\mathbf{C}^n$ is the same as a bundle on the sphere $S^{2n}$ with a trivialization on a ball, and this itself can be described by a map from the equatorial $S^{2n-1}$ to $GL(m, \mathbf{C})$ relating the two trivializations. Since the definition of the K-group involves isomorphism classes of bundles and adding on trivial bundles, $K_c(\mathbf{C}^n)$ is isomorphic to homotopy classes of maps $S^{2n-1} \to U(m)$ for $m$ large. This is the homotopy group $\pi_{2n-1}(U(m))$ which Bott showed was infinite cyclic (and $\pi_{2n}(U(m)) = 0$).

If we consider instead real vector bundles on a space $X$ then one defines K-groups $KO(X)$ and then $KO_c(\mathbf{R}^n)$ is defined by the homotopy group $\pi_{n-1}(O(m))$. In this case Bott had showed that $\pi_{n-1}(O(m))$ is eightfold periodic in $n$ and $\pi_i(O(m))$ for $i = 0, 1, 2, \ldots, 7$ mod 8 is

$$\mathbf{Z}_2 \quad \mathbf{Z}_2 \quad 0 \quad \mathbf{Z} \quad 0 \quad 0 \quad 0 \quad \mathbf{Z}.$$

The K-theory definition of the symbol class above then shows that a real elliptic operator $D$ in dimensions $4k$ has an integer invariant (which up to a multiple is just the ordinary index) but in dimensions $8k+1$ and $8k+2$ there is an invariant in $\mathbf{Z}_2$, and the challenge is to interpret this analytically.

The answer lies with skew-adjoint real Fredholm operators. By skew-adjointness the kernel and cokernel have the same dimension so the ordinary index vanishes, but the dimension of the kernel modulo 2 is a deformation invariant. For elliptic differential operators we can see simple examples of this even on the circle (dimension 1 mod 8!). There are two real line bundles over the circle—the trivial bundle and the Möbius band. The operator

$$D = \frac{d}{d\theta}$$

with periodic boundary conditions is a skew-adjoint operator on the trivial bundle and has a 1-dimensional kernel, the constants. With anti-periodic conditions the line bundle is the Möbius band and the operator has kernel zero. Slightly more interesting is a skew adjoint third order operator on the trivial bundle over $S^1$:

$$D = \frac{d^3}{d\theta^3} + 2u\frac{d}{d\theta} + u'.$$

This has a one or three-dimensional space of solutions but never a two-dimensional one.

The mod 2 index theorem in Part V of the papers of Atiyah and Singer derives the dimension modulo 2 of the kernel of a real skew-adjoint elliptic operator in terms of its symbol class.

In $8k+1$-dimensions the spin representation is real and the Dirac operator skew-adjoint so this gives a $\mathbf{Z}_2$-invariant for spin manifolds in this dimension. In $8k+2$ dimensions the Dirac operator can be considered as a skew-adjoint complex anti-linear operator and the mod 2 invariant is the complex dimension mod 2 of the kernel.

**5.2. Theta Characteristics.** Applications of the mod 2 theorem are not so numerous as the other versions but in [11] Atiyah revisits some classical results on Riemann surfaces with this new tool. On a Riemann surface $M$ a spin-structure is defined by a holomorphic square root $K^{1/2}$ of the canonical bundle and the Dirac operator is

$$\bar{\partial} : C^{\infty}(K^{1/2}) \to C^{\infty}(K^{1/2}\bar{K}).$$

A metric identifies $\bar{K}$ with $K^*$ and so

$$K^{1/2}\bar{K} \cong K^{-1/2} \cong \bar{K}^{1/2}.$$

This is the identification of the two spinor bundles which makes the Dirac operator antilinear.

The null space of $\bar{\partial}$ is the space of holomorphic sections of $K^{1/2}$ and the $\mathbf{Z}_2$-invariant is the dimension modulo 2 of this space.

Any two square roots differ by a holomorphic line bundle $L$ such that $L^2$ is holomorphically trivial so there are $2^{2g}$ such choices where $g$ is the genus of $M$. These are the different spin structures referred to in Sect. 2.6. The invariant is zero for $2^{g-1}(2^g+1)$ of these and 1 for the other $2^{g-1}(2^g-1)$. Thus for an elliptic curve where $K$ is trivial there is $2^{1-1}(2^1-1)=1$ square root, the trivial one, which has an odd (namely one) number of sections. For a quartic plane curve there are $2^{3-1}(2^3-1)=28$ square roots with an odd (one again) number of sections and these are the celebrated 28 bitangents. Classically these square roots are known as theta characteristics and they have odd or even type, but the 19th century arguments involved the zeros of the Riemann theta function whereas the derivation in [11] uses elementary properties of the group $KO(M)$. One amusing result is that a real quartic with no real points has exactly four real bitangents.

The earlier example of $d/d\theta$ on the circle is precisely the Dirac operator and the two real line bundles two spin structures where the dimension mod 2 of the kernel distinguishes them.

**5.3. Positive Scalar Curvature.** The Lichnerowicz vanishing theorem in Sect. 3.3 tells us that if a spin manifold of whatever dimension admits a metric of positive scalar curvature then the kernel of the Dirac operator is zero. In dimensions $8k+1, 8k+2$ this means the mod 2 index vanishes. Surprisingly there are exotic spheres (manifolds homotopically equivalent to a sphere) in these dimensions for which the invariant is known to be non-zero. We deduce that these spheres cannot have metrics of positive scalar curvature.

Perhaps the best way to describe the invariant is to say that it is a spin cobordism invariant. Thom's cobordism theory can be modified to put extra structure on the manifolds in question. It was oriented cobordism that gave the Pontryagin numbers that Hirzebruch used for his proof of the signature theorem. These exotic spheres have the property that, while they are themselves spin manifolds, and while they bound an oriented manifold, they do not bound a spin manifold.

The spin-cobordism interpretation of obstructions to positive scalar curvature led Gromov and Lawson [23] to ask whether these were the only obstructions. They showed that any manifold which can be obtained from one of positive scalar curvature by performing surgery in codimension greater than 2 also carries a metric of positive scalar curvature. Surgery is a process which operates within a cobordism class and as a consequence they deduced that any compact simply connected spin manifold of dimension $\geq 5$ which is spin-cobordant to a manifold of positive scalar curvature also carries a metric of positive scalar curvature. Somewhat later, by using techniques from stable homotopy theory to analyze spin cobordism in more detail, Stolz [29] succeeded in proving that these invariants—the $\hat{A}$ genus in dimension $4k$ and the mod 2 index in dimensions $8k+1, 8k+2$ are the only obstructions for a simply-connected manifold to have positive scalar curvature.

## 6 The Index for Families

**6.1. Fredholm Operators.** The ordinary integer index is a deformation invariant of a Fredholm operator. This was the starting point for the index theorem. But the space $\mathcal{F}$ of all Fredholm operators on a fixed Hilbert space contains more topological information than that. A family of Fredholm operators parametrized by a space $X$ is a continuous map $f\colon X \to \mathcal{F}$ and we can consider the set of homotopy classes $[X, \mathcal{F}]$ of such maps. A theorem of Atiyah (and independently K. Jänich) says that

$$[X, \mathcal{F}] \cong K(X).$$

When $X$ is a point $[pt., \mathcal{F}] \cong K(pt.) \cong \mathbf{Z}$ is the set of components so we learn that the components of $\mathcal{F}$ are determined by the index. If $A, B$ are two Fredholm operators then

$$\dim \ker AB \leq \dim \ker A + \dim \ker B$$

so $AB$ has finite dimensional kernel and using adjoints the same is true of cokernels. This product induces the product on $[X, \mathcal{F}]$.

If $X$ is connected and the kernel of $f(x) = A_x$ has constant rank $m$, then since the index is constant the cokernel has constant rank $(m - \operatorname{ind} A_x)$ and, as $x$ varies over $X$, we have two vector bundles over $X$, whose difference $\ker A - \operatorname{coker} A$ clearly defines a class in $K(X)$. This is the basis of the isomorphism above but the key issue is that the map extends even to the case where the dimension jumps.

Suppose now that $Z \xrightarrow{\pi} X$ is a smooth fibre bundle whose fibre is diffeomorphic to a compact manifold $M^n$, and suppose we have vector bundles $V_+$, $V_-$ over $Z$ and for each $x \in X$ a smoothly varying elliptic operator

$$D : C^\infty(Z_x, V_+|_{Z_x}) \to C^\infty(Z_x, V_-|_{Z_x}).$$

Then we can convert this into a family of Fredholm operators and get an element

$$\operatorname{ind} D \in K(X).$$

The index theorem for families, proved in Part IV of the Atiyah–Singer papers, expresses this analytical class in terms of a topological class defined by the family of symbols.

If $Z \xrightarrow{\pi} X$ is a holomorphic fibration and $E$ is a holomorphic vector bundle on $Z$, then the $\bar\partial$ elliptic complex along the fibres is an example. The Grothendieck–Riemann–Roch theorem for this is then an example of the index theorem. The sheaf $\mathcal{O}(E)$ of holomorphic sections of $E$ on $Z$ defines coherent sheaves over $X$ whose sections over an open set $U \subset X$ are $H^p(\pi^{-1}(U), \mathcal{O}(E))$. The alternating sum of these defines an element $\pi_! \mathcal{O}(E)$ in the Grothendieck group of coherent sheaves on $X$ which maps under the Chern character to the cohomology of $X$. Then

$$\operatorname{ch}\big(\pi_! \mathcal{O}(E)\big) \operatorname{td}(TX) = \pi_*\big(\operatorname{ch}(E) \operatorname{td}(TZ)\big)$$

where $\pi_*$ is the map defined by integration over the fibres $\pi_* : H^p(M, \mathbf{R}) \to H^{p-n}(M, \mathbf{R})$.

The cohomological version of the index theorem for families has a similar form. If $D$ is a family of elliptic operators, it has a symbol class $[\sigma] \in K_c(TZ)$ and an analytical index $\operatorname{ind} D \in K(X)$. Then

$$\operatorname{ch}(\operatorname{ind} D) = (-1)^n \pi_*\big(\operatorname{ch}\sigma \operatorname{td}(TZ \otimes \mathbf{C})\big)$$

where $\pi_* \colon H_c^*(TZ) \to H^*(X)$ is integration over the fibres. This is an important formula for calculations but the problem is still best framed in K-theoretical terms. In particular the mod 2 index has a family version.

**6.2. Jumping of Dimension.** We have noted already that in a continuous family of Fredholm operators while the integer index remains constant the dimension of the kernel may jump. The index theorem for families can sometimes detect this. The integer function $\dim\ker A$ is upper semi-continuous and so the jumps are upwards in dimension. Suppose that we have a family where the index is zero. Then if $\dim\ker A$ is always zero in a family, so is $\dim\operatorname{coker} A$ and the K-theory index in $K(X)$ vanishes. Hence if we know the index is non-zero there must be non-trivial jumps somewhere in the family.

One classical example is to take the Dirac operator on a Riemann surface $M$

$$\bar\partial \colon C^\infty(K^{1/2} \otimes L) \to C^\infty(K^{1/2} \otimes L\bar{K})$$

with coefficient bundle a line bundle with flat unitary connection. The index is zero here since the Riemann–Roch formula gives $g-1+1-g=0$. The flat line bundles are parametrized by the torus $X = H^1(M, \mathbf{R}/\mathbf{Z})$ and if we choose a universal line bundle $L$ over $Z = M \times X$, we have a setting to apply the index theorem and find a class in $K(X)$, or from the Chern character in $H^*(X)$. This is a holomorphic situation so it is actually the Grothendieck–Riemann–Roch theorem we use.

The universal bundle $L$ has Chern class

$$c(L) = \sum_1^g (x_i y_i' - y_i x_i') \in H^1(M, \mathbf{Z}) \otimes H^1(X, \mathbf{Z}) \subset H^2(M \times X, \mathbf{Z})$$

where $x_1, \ldots, x_g, y_1, \ldots, y_g$ is a symplectic basis of $H^1(M, \mathbf{Z})$ and we use the isomorphism $H^1(X, \mathbf{Z}) \cong H^1(M, \mathbf{Z})$ to define a corresponding basis $x_1', \ldots, x_g'$, $y_1', \ldots, y_g'$. The G–R–R formula is then

$$\mathrm{ch}\big(\pi_! \mathcal{O}(L \otimes K^{1/2})\big)\,\mathrm{td}(TX) = \pi_*\big(\mathrm{ch}(L \otimes K^{1/2})\,\mathrm{td}(TX)\,\mathrm{td}(M)\big)$$

or, since $TX$ is trivial,

$$\begin{aligned}\mathrm{ch}\big(\pi_! \mathcal{O}(L)\big) &= \pi_*\left(\left(1 + c(L) - \frac{1}{2}c_1(TM) + \frac{1}{2}c(L)^2\right)\left(1 + \frac{1}{2}c_1(TM)\right)\right)\\ &= \pi_*\left(1 + c(L) + \frac{1}{2}c(L)^2\right)\end{aligned}$$

since $H^p(M)$ vanishes for $p \geq 2$. Now $c(L)$ has no component in $H^2(M) \otimes H^0(X)$ so integrating over $M$ kills this. We have

$$c(L)^2 = \left(\sum_1^g (x_i y_i' - y_i x_i')\right)^2 = -2\omega\theta$$

where $\omega = x_1 y_1 = x_2 y_2 = \cdots = x_n y_n$ is the generator of $H^2(M, \mathbf{Z})$ and $\theta = \sum_1^g x_i' y_i'$. It follows that

$$\mathrm{ch}\big(\pi_! \mathcal{O}(L)\big) = -\theta.$$

This is non-zero and so the dimension of the kernel jumps. It does so of course on the theta divisor in the torus $X$, which is Poincaré dual to the cohomology class $\theta$.

This is a classical example but note that $\theta$ is non-zero even when $X$ has genus one i.e. is a torus itself. So even when base and fibre of $Z$ have no non-trivial characteristic classes there is still a non-trivial index. In higher dimensions one can do the same with $M$ an even-dimensional torus $T^{2m} = \mathbf{R}^{2m}/\mathbf{Z}^{2m}$. The one-form $\sum y_i dx_i$ describes a family of flat connections on the trivial bundle over $T^{2m}$ parametrized by $(y_1, \ldots, y_{2m}) \in \mathbf{R}^{2m}/2\pi\mathbf{Z}^{2m} = X$. The curvature of this line bundle $L$ over $T^{2m} \times X$ is

$$F = d\sum y_i dx_i = \sum dy_i \wedge dx_i.$$

The Chern character now contributes a non-trivial term to the index by integrating $F^{2m}$ over the fibre to give a non-zero class in $H^{2m}(X)$. This means in particular that there is a non-trivial jump in the dimension of the kernel of the Dirac operator. In particular the torus cannot have a metric of positive scalar curvature since in Lichnerowicz's formula there is no contribution from the zero curvature of the line bundle. This result, due to Gromov and Lawson, has spurred a great deal of work understanding which fundamental groups are compatible with positive scalar curvature. They all involve indices of a more sophisticated nature, taking values in the K-theory of the $C^*$ algebra associated to a discrete group.

## 7 The Local Index Theorem

**7.1. The Heat Kernel.** The foundational papers of Atiyah and Singer on the index theorem expressed the analytic index as a topological invariant. It could be represented in different ways depending on which version of cohomology one used and how one represented characteristic classes. However, the main operators of interest such as the Dirac operator were expressed in terms of a Riemannian metric. In this setting there was also a natural way to represent the characteristic classes—by using the curvature of the Levi-Civita connection. In the early 1970s new proofs emerged which capitalized on this fact and provided the tools for the study of another range of index problems. The original idea came from work of McKean and Singer.

Suppose $D$ is a first order elliptic operator such as the Dirac operator, then one considers the self-adjoint operators $DD^*$ and $D^*D$. They have, on a compact manifold, a discrete spectrum $0 \leq \lambda_0 \leq \lambda_1 \dots$ each value taken only a finite number of times. If $\phi_i$ are the eigenvectors, then the *heat kernel*

$$H(x, y, t) = \sum_{j=0}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y)$$

is for $t > 0$ a well-behaved smooth function, formally written as $e^{-tD^*D}$ or $e^{-tDD^*}$. In particular it has a trace

$$\operatorname{tr} e^{-tD^*D} = \sum_0^{\infty} e^{-\lambda_j t} \qquad \operatorname{tr} e^{-tDD^*} = \sum_0^{\infty} e^{-\mu_j t}.$$

Now if $D^*D\phi = \lambda\phi$, then $DD^*D\phi = \lambda D\phi$ so that if $D\phi$ is non-zero, then it is an eigenvector for $DD^*$. It follows that the non-zero eigenvalues of $DD^*$ and $D^*D$ are the same so that

$$\operatorname{tr} e^{-tD^*D} - \operatorname{tr} e^{-tDD^*} = \dim\ker D^*D - \dim\ker DD^* = \operatorname{ind} D.$$

In particular, this expression is independent of $t$ and so one can consider the behaviour of each term on the left hand side as $t$ approaches 0.

In this case, along the diagonal one has an asymptotic expansion

$$H(x,x,t) \sim \sum_{j=0}^{\infty} a_j t^{-n/2+j}$$

where the coefficients $a_j$ are determined locally. In other words each one is an algebraic expression in a finite number of derivatives of the coefficients of the operator $D$—in the case of the Dirac operator, the Riemannian metric by which it is defined. Then

$$\begin{aligned}\operatorname{ind} D &= \int_M \left[ \sum_{j=0}^{\infty} a_j(D^*D)t^{-n/2+j} - \sum_{j=0}^{\infty} a_j(DD^*)t^{-n/2+j} \right] \\ &= \int_M \big( a_{n/2}(D^*D) - a_{n/2}(DD^*) \big).\end{aligned}$$

(Note that this already implies that the index is zero in odd dimensions.)

The first term in the asymptotic expansion involves simply the volume, but the relevant terms for the index theorem are much further along and in principle could involve many derivatives of the metric. Evidence that a proof of the index theorem could be produced like this came from the ingenious cancellations that the young Indian mathematician Vijay Patodi used to prove the Gauss–Bonnet theorem with the same approach. Then appeared the work of Gilkey which led to a completely new proof of the index theorem. This involved: characterizing certain essential features of the polynomials, using invariant theory of the orthogonal group; showing that this yielded the Pontryagin forms defined from the Levi-Civita connection; finally, much as in Hirzebruch's signature theorem, evaluating on standard examples to get the correct coefficients.

Proving the index theorem this way is enough to get the general integer index theorem because the symbols of the Dirac operator with all possible coefficient bundles generate all the homotopy classes. It raised other questions of a different nature however, in the complex case for example. The use of Riemannian methods meant that Kähler manifolds could be treated this way, but how did one get a local Riemann–Roch theorem for a general complex manifold where the Riemannian connection is not compatible with the complex structure? Bismut [15] discovered that to get a local index theorem one has to use a Riemannian connection whose torsion tensor is defined by a closed 3-form $H$, thought of as a 1-form with values in skew-adjoint endomorphisms of the tangent bundle. In other words one uses connections of the form

$$\nabla + H$$

where $\nabla$ is the usual Levi-Civita connection, which has zero torsion. The curious feature here is that the local index formula for the Dirac operator defined by the connection $\nabla + H/3$ involves the Pontryagin forms of the connection $\nabla - H$. Moreover in the Hermitian case if the connection $\nabla + H$ preserves the complex structure, the $\bar\partial + \bar\partial^*$ operator is defined using $\nabla + H/3$. In the context of Lie groups this was called by Kostant the *cubic Dirac operator* and has some special features, notably that the zero-order term in the Lichnerowicz formula is still a scalar function.

**7.2. The Eta Invariant.** One of the areas which Atiyah, Singer and Patodi developed using heat equation methods was the geometrical study of some boundary value problems. The signature theorem was one motivation for this. We saw in Sect. 2.4 how the middle dimensional cohomology of a compact oriented $4k$-dimensional manifold has a non-degenerate symmetric bilinear form, which allows the definition of the signature. For a $4k$-dimensional manifold $M$ with boundary $\partial M$, one can also define the signature, using compactly supported closed forms. The additivity theorem of Novikov asserts that when two compact oriented $4k$-manifolds are glued by an orientation reversing diffeomorphism of their boundaries, the signature of their union is the sum of their signatures.

On a compact manifold, the signature is given by the integral of a differential form given as a polynomial in Pontryagin forms by Hirzebruch's formula. If we do this for the manifold with boundary $M$, this will not necessarily be so. However, if the metric near the boundary is a product we can smoothly glue together two such manifolds to get a Riemannian manifold, and it follows from Novikov additivity that the difference between the signature of $M$ and the integral is an invariant only of the Riemannian metric on the $4k-1$-dimensional boundary $\partial M$. Identifying this invariant led to another type of index theorem. As usual, it is easiest to describe for the basic Dirac operator $D$.

In dimension $4k-1$ the Dirac operator is real and self-adjoint so it has real eigenvalues $\lambda_i$, both positive and negative since $D$ is a first-order operator. One then defines

$$\eta(s) = \sum_{\lambda_j \neq 0} (\operatorname{sgn} \lambda_j)|\lambda_j|^{-s}.$$

This is holomorphic when the real part of $s$ is large but has a meromorphic extension to the whole complex plane and is finite at $s = 0$. Formally speaking then, $\eta(0)$ is the difference between the (infinite) number of positive and negative eigenvalues of $D$—the "signature" of the quadratic form $(D\varphi, \varphi)$.

This eta invariant appears as a correction term in the formula for the index of a Fredholm operator for the manifold with boundary $M$. Atiyah, Patodi and Singer consider the Dirac operator on $M$

$$D : C^\infty(S_+) \to C^\infty(S_-)$$

and solutions to $D\varphi = 0$ with the boundary condition that the projection of $\varphi$ onto the space spanned by the positive eigenvectors of the Dirac operator on the boundary is zero. It turns out that this is Fredholm and there is an index theorem of the form

$$\operatorname{ind} D = \int_M \hat{A}(TM) - \frac{1}{2}\big(\eta(0) + h\big)$$

where $h$ is the dimension of the kernel of the Dirac operator on the boundary.

Another way of interpreting this result is to note that if the projection onto the positive part and zero eigenspace vanishes, then on the boundary $\varphi$ has an expansion

$$\varphi = \sum_{\lambda_j < 0} c_j \phi_j$$

and then

$$\sum_{\lambda_j < 0} e^{\lambda_j t} c_j \phi_j$$

decays exponentially and is an $\mathcal{L}^2$ solution to the Dirac equation $D\varphi = 0$ on the cylinder $\partial M \times [0, \infty)$. Thus the null-space is the space of $\mathcal{L}^2$ solutions to $D\varphi = 0$ on the non-compact manifold obtained by glueing the cylinder to $M$ at its boundary. Replacing the Dirac operator by the signature operator and linking compactly supported cohomology with $\mathcal{L}^2$-cohomology gives the signature formula:

$$\tau(M) = \int_M L(TM) + (-1)^{k+1}\eta(0).$$

**7.3. Quantum Field Theory.** It turned out that, unknown to Atiyah, Singer and Patodi, the development of the local index theorem by mathematicians coincided with an interest in the theorem from theoretical physicists. As Singer has remarked, some of this was taking place in offices around the corner from his own in MIT. The context in 1970 was the *chiral anomaly* of Roman Jackiw. An anomalous symmetry in quantum field theory is a symmetry of the action, but not of the measure. In the standard model of electroweak interactions the classical current conservation law $\partial_\mu J^B_\mu = 0$ is replaced by

$$\partial_\mu J^B_\mu = \frac{g^2 C}{32\pi^2} \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}.$$

The right hand side here is essentially the second Chern form for the connection defined by the gauge theory. Moreover an important physical fact is that this term is a total derivative involving $\partial_\mu K_\mu$ where

$$K_\mu = 2\epsilon_{\mu\nu\alpha\beta}\left(A_\nu \partial_\alpha A_\beta + \frac{2}{3} i g A_\nu A_\alpha A_\beta\right).$$

Mathematically this term only makes sense having chosen a trivialization of the bundle—a choice of gauge—so that the connection is $\partial_\mu + gA_\mu$ but this so-called Chern–Simons expression appears naturally in the Atiyah–Singer–Patodi formula. On a 3-manifold where the bundle is globally trivial the integral of this expression is well-defined modulo the integers and the eta-invariant is a real lift of it.

In terms of methodology, the physicists were happy with heat kernels but at that stage knew little about the topology. The ingredients for studying anomalies were the same as for the index theorem and it turned out that certain anomalies were precisely indices. It was Singer's interest in this parallel evolution that led him to talk more to the physicists and subsequently to introduce the problem of Yang–Mills instantons to Atiyah and coworkers when he visited Oxford in 1977.

Another interface with physics came from supersymmetric field theories consisting of a physical system described by a Hamiltonian $H$ and two *supercharges* $Q$ and $Q^\dagger$ which map fermions to bosons and vice versa. They satisfy the anti-commutation relation $\{Q, Q^\dagger\} = H$ so that both supercharges commute with $H$ and satisfy $Q^2 = (Q^\dagger)^2 = 0$. Clearly there is an example given by $Q = d : \Omega^{even} \to \Omega^{odd}$, $Q^\dagger = d^*$ and $H = dd^* + d^*d$, the Hodge Laplacian. More generally any elliptic complex fits this scheme, and the index theorem becomes the problem of evaluating the so-called Witten index.

This new viewpoint led to supersymmetric proofs of the index theorem by physicists [1, 18] which were given a rigorous mathematical form by Getzler [14, 21].

**7.4. The Supersymmetric Proof.** The physics motivation for Getzler's proof is the background expansion used in the supersymmetric path integral to obtain the small fluctuation Lagrangian. One uses the Dirac operator with coefficient bundle, normal coordinates at a point and an expansion $x_i + \sqrt{t}\,y_i$. Then Getzler introduces a clever rescaling including that of the Clifford algebra, so that if the degree of $t$ is 2, of $x_i$ is one, then the degree of $e_i$, a generator of the Clifford algebra is $-1$. The effect is that as $t \to 0$ the Clifford algebra approaches the Grassmann algebra and the Dirac Laplacian approaches

$$\left(\partial_i - \frac{1}{4}R_{ij}x_j^\wedge\right)^2 + F\wedge$$

where $R_{ij}$ is the Riemann curvature tensor considered as a matrix of 2-forms. The fact that the Lichnerowicz formula involves just the scalar curvature and doesn't contribute other terms in the Clifford algebra is a key point here, and explains the presence of the cubic Dirac operator in Bismut's modification [15].

The heat kernel is then approximated by the heat kernel for the harmonic oscillator

$$-\Delta + \theta_{ij}x_i x_j$$

but using exterior multiplication instead of scalar multiplication of functions. The heat kernel for the $m$-dimensional harmonic oscillator is

$$(4\pi t)^{-m/2}\det\left[\frac{2t\sqrt{\theta}}{\sinh 2t\sqrt{\theta}}\right]$$
$$\times \exp -\frac{1}{4t}\left[\left(\frac{2t\sqrt{\theta}}{\tanh 2t\sqrt{\theta}}\right)_{ij}(x_i x_j + y_i y_j) - 2\left(\frac{2t\sqrt{\theta}}{\sinh 2t\sqrt{\theta}}\right)_{ij} x_i y_j\right].$$

Replacing $\theta_{ij}$ by the matrix of forms $R_{ij}$ leads to the index formula. The rescaling has the effect that the index term $a_{n/2}$ in the asymptotic expansion becomes the leading coefficient.

The obvious feature of this formula is the natural presence of the expression

$$\frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}$$

—the polynomial defining the $\hat{A}$-genus which prompted Atiyah's original question to Singer in 1962. The physics thus provides some form of explanation of the role of these very special polynomials.

## References

1. Alvarez-Gaumé, L.: Supersymmetry and the Atiyah–Singer index theorem. Commun. Math. Phys. **90**, 161–173 (1983)
2. Atiyah, M.: Mathematician, http://www.peoplesarchive.com
3. Atiyah, M.F., Singer, I.M.: The index of elliptic operators I. Ann. Math. **87**, 484–530 (1968)
4. Atiyah, M.F., Singer, I.M.: The index of elliptic operators III. Ann. Math. **87**, 546–604 (1968)
5. Atiyah, M.F., Singer, I.M.: The index of elliptic operators IV. Ann. Math. **93**, 119–138 (1971)
6. Atiyah, M.F., Singer, I.M.: The index of elliptic operators V. Ann. Math. **93**, 139–149 (1971)
7. Atiyah, M.F., Singer, I.M., Segal, G.B.: The index of elliptic operators II. Ann. Math. **87**, 531–545 (1968)
8. Atiyah, M.F., Bott, R.: A Lefschetz fixed point formula for elliptic complexes I. Ann. Math. **86**, 374–407 (1967)
9. Atiyah, M.F., Bott, R.: A Lefschetz fixed point formula for elliptic complexes II Applications. Ann. Math. **88**, 451–491 (1968)
10. Atiyah, M.F., Hirzebruch, F.: Spin-manifolds and group actions. In: Haefliger, A., Narasimhan, R. (eds.) Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), pp. 18–28. Springer, New York (1970)
11. Atiyah, M.F.: Riemann surfaces and spin structures. Ann. Sci. École Norm. Sup. **4**, 47–62 (1971)
12. Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Deformations of instantons. Proc. Nat. Acad. Sci. U.S.A. **74**, 2662–2663 (1977)
13. Atiyah, M.F., Hitchin, N.J., Drinfeld, V.G., Manin, Yu.I.: Construction of instantons. Phys. Lett. A **65**, 185–187 (1978)
14. Berline, N., Getzler, E., Vergne, M.: Heat Kernels and Dirac Operators. Springer, Berlin (1992)
15. Bismut, J.-M.: A local index theorem for non-Kähler manifolds. Math. Ann. **284**, 681–699 (1989)
16. Bott, R., Taubes, C.: On the rigidity theorems of Witten. J. Am. Math. Soc. **2**, 137–186 (1989)
17. Donaldson, S.K., Kronheimer, P.B.: The Geometry of Four-Manifolds. Oxford Univ. Press, Oxford (1990)
18. Friedan, D., Windey, P.: Supersymmetric derivation of the Atiyah–Singer index and the chiral anomaly. Nucl. Phys. B **235**, 395–416 (1984)
19. Gelfand, I.M.: On elliptic equations. (Russ.) Usp. Mat. Nauk **15**, 121–132 (1960)
20. Gelfand, I.M.: On elliptic equations. Russ. Math. Surv. **15**, 113–123 (1960)
21. Getzler, E.: A short proof of the local Atiyah–Singer index theorem. Topology **25**, 111–117 (1986)
22. Gray, J.J.: The Riemann–Roch theorem and geometry, 1854–1914. In: Proceedings of the International Congress of Mathematicians, vol. III, Berlin (1998). Doc. Math. 1998, Extra vol. III, 811–822 (electronic)
23. Gromov, M., Lawson, H.B. Jr.: The classification of simply connected manifolds of positive scalar curvature. Ann. Math. **111**, 423–434 (1980)

24. Hirzebruch, F.: Neue topologische Methoden in der algebraischen Geometrie. Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 9. Springer, Berlin (1956)
25. Hirzebruch, F.: The signature theorem: reminiscences and recreation. In: Prospects in Mathematics. Ann. Math. Stud., vol. 70, pp. 3–31. Princeton Univ. Press, Princeton (1971)
26. Lichnerowicz, A.: Spineurs harmoniques. C. R. Acad. Sci. Paris **257**, 7–9 (1963)
27. Palais, R.S.: Seminar on the Atiyah–Singer Index Theorem. Ann. Math. Stud., vol. 57. Princeton Univ. Press, Princeton (1965)
28. Singer, I.M.: Letter to Michael. In: Yau, S.-T. (ed.) The Founders of Index Theory: Reminiscences of Atiyah, Bott, Hirzebruch, and Singer, pp. 296–297. International Press, Somerville (2003)
29. Stolz, S.: Simply connected manifolds of positive scalar curvature. Bull. Am. Math. Soc. **23**, 427–432 (1990)

# List of Publications for Sir Michael Atiyah


## 1952

[1] A note on the tangents of a twisted cubic. *Proc. Cambridge Philos. Soc.*, 48:204–205.

## 1954

[2] (with W. Hodge). Formes de seconde espèce sur une variété algébrique. *C. R. Acad. Sci. Paris*, 239:1333–1335.

## 1955

[3] Complex fibre bundles and ruled surfaces. *Proc. London Math. Soc. (3)*, 5:407–434.
[4] (with W. Hodge). Integrals of the second kind on an algebraic variety. *Ann. of Math. (2)*, 62:56–91.

## 1956

[5] On the Krull–Schmidt theorem with application to sheaves. *Bull. Soc. Math. France*, 84:307–317.

## 1957

[6] Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)*, 7:414–452.
[7] Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85:181–207.

## 1958

[8] Some examples of complex manifolds. *Bonn. Math. Schr.* no. 6, 28 pp.
[9] Complex analytic connections in fibre bundles. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 77–82. Universidad Nacional Autónoma de México and UNESCO, Mexico City.
[10] On analytic surfaces with double points. *Proc. Roy. Soc. London. Ser. A*, 247:237–244.

## 1959

[11] (with F. Hirzebruch). Quelques théorèmes de non-plongement pour les variétés différentiables. *Bull. Soc. Math. France*, 87:383–396.
[12] (with F. Hirzebruch). Riemann–Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.*, 65:276–281.


H. Holden, R. Piene (eds.), *The Abel Prize*,

DOI 10.1007/978-3-642-01373-7_15, 

## 1960

[13] (with J.A. Todd). On complex Stiefel manifolds. *Proc. Cambridge Philos. Soc.*, 56:342–353.

## 1961

[14] (with F. Hirzebruch). Cohomologie-Operationen und charakteristische Klassen. *Math. Z.*, 77:149–187.

[15] (with F. Hirzebruch). Charakteristische Klassen und Anwendungen. *Enseignement Math. (2)*, 7:188–213 (1962).

[16] Characters and cohomology of finite groups. *Inst. Hautes Études Sci. Publ. Math.*, (9):23–64.

[17] (with F. Hirzebruch). Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, R.I.

[18] Thom complexes. *Proc. London Math. Soc. (3)*, 11:291–310.

[19] Bordism and cobordism. *Proc. Cambridge Philos. Soc.*, 57:200–208.

[20] (with F. Hirzebruch). Bott periodicity and the parallelizability of the spheres. *Proc. Cambridge Philos. Soc.*, 57:223–226.

## 1962

[21] Vector bundles and the Künneth formula. *Topology*, 1:245–248.

[22] (with F. Hirzebruch). The Riemann–Roch theorem for analytic embeddings. *Topology*, 1:151–166.

[23] (with F. Hirzebruch). Analytic cycles on complex manifolds. *Topology*, 1:25–45.

[24] Immersions and imbeddings of manifolds. *Topology*, 1:125–132.

## 1963

[25] The Grothendieck ring in geometry and topology. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, 442–446. Inst. Mittag-Leffler, Djursholm.

[26] (with I.M. Singer). The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69:422–433.

## 1964

[27] (with R. Bott). The index problem for manifolds with boundary. In *Differential Analysis, Bombay Colloq., 1964*, pages 175–186. Oxford Univ. Press, London.

[28] (with R. Bott). On the periodicity theorem for complex vector bundles. *Acta Math.*, 112:229–247.

[29] (with R. Bott and A. Shapiro). Clifford modules. *Topology*, 3(suppl. 1):3–38.

## 1965

[30] R.S. Palais. *Seminar on the Atiyah–Singer index theorem*. With contributions by M.F. Atiyah, A. Borel, E.E. Floyd, R.T. Seeley, W. Shih and R. Solovay. Annals of Mathematics Studies, No. 57. Princeton University Press, Princeton, N.J. Also available in Russian, Izdat. “Mir”, Moscow, 1970.

[31] On the $K$-theory of compact Lie groups. *Topology*, 4:95–99.

## 1966

[32] A Lefschetz fixed point formula for holomorphic mappings. In *Contemporary Problems in Theory Anal. Functions (Internat. Conf., Erevan, 1965) (Russian)*, pages 28–32. Izdat. "Nauka", Moscow.
[33] $K$-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386.
[34] Power operations in $K$-theory. *Quart. J. Math. Oxford Ser. (2)*, 17:165–193.
[35] (with J.F. Adams). $K$-theory and the Hopf invariant. *Quart. J. Math. Oxford Ser. (2)*, 17:31–38.
[36] (with R. Bott). A Lefschetz fixed point formula for elliptic differential operators. *Bull. Amer. Math. Soc.*, 72:245–250.
[37] The role of algebraic topology in mathematics. *J. London Math. Soc.*, 41:63–69. Also available in *Geometrie*, pages 74–83, Wege der Forschung, Band CLXXVII, Wissenschaftliche Buchgesellschaft, Darmstadt, 1972 (in German).

## 1967

[38] *$K$-theory*. Lecture notes by D.W. Anderson. W.A. Benjamin, Inc., New York–Amsterdam. Also available in Russian, Izdat. "Mir", Moscow. Second edition, Advanced Book Classics, Addison-Wesley Publishing Company, Redwood City, CA, 1989.
[39] (with C.T.C. Wall). Cohomology of groups. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 94–115. Thompson, Washington, D.C.
[40] (with R. Bott). A Lefschetz fixed point formula for elliptic complexes. I. *Ann. of Math. (2)*, 86:374–407.
[41] Algebraic topology and elliptic operators. *Comm. Pure Appl. Math.*, 20:237–249.

## 1968

[42] (with I.M. Singer). The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530. Also in *Uspehi Mat. Nauk*, 23(5 (143)):99–142.
[43] (with G.B. Segal). The index of elliptic operators. II. *Ann. of Math. (2)*, 87:531–545. Also in *Uspehi Mat. Nauk*, 23(6 (144)):135–149.
[44] (with I.M. Singer). The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604. Also in *Uspehi Mat. Nauk*, 24(1 (145)):127–182, 1969.
[45] Global aspects of the theory of elliptic differential operators. In *Proc. Internat. Congr. Math. (Moscow, 1966)*, pages 57–64. Izdat. "Mir", Moscow.
[46] (with R. Bott). A Lefschetz fixed point formula for elliptic complexes. II. Applications. *Ann. of Math. (2)*, 88:451–491.
[47] Bott periodicity and the index of elliptic operators. *Quart. J. Math. Oxford Ser. (2)*, 19:113–140.

## 1969

[48] (with I.M. Singer). Index theory for skew-adjoint Fredholm operators. *Inst. Hautes Études Sci. Publ. Math.*, (37):5–26.
[49] (with G.B. Segal). Equivariant $K$-theory and completion. *J. Differential Geometry*, 3:1–18.
[50] The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84. Univ. Tokyo Press, Tokyo.
[51] Algebraic topology and operators in Hilbert space. In *Lectures in Modern Analysis and Applications. I*, pages 101–121. Springer, Berlin.
[52] (with D.O. Tall). Group representations, $\lambda$-rings and the $J$-homomorphism. *Topology*, 8:253–297.
[53] (with I.G. Macdonald). *Introduction to commutative algebra*. Addison–Wesley Publishing Co., Reading, Mass.–London–Don Mills, Ont. Also available in Russian, Izdat. "Mir", Moscow, 1972.
[54] Wandel und Fortschritt in der Mathematik. *Blid. Wiss.*, 6:314–323. Also in *Mathematiker über die Mathematik*, pages 202–218, Springer-Verlag, Berlin, 1974.

## 1970

[55] (with R. Bott and L. Gårding). Lacunas for hyperbolic differential operators with constant coefficients. I. *Acta Math.*, 124:109–189. Also available in *Uspehi Mat. Nauk*, 26(2(158)):25–100, 1971 (in Russian).

[56] (with F. Hirzebruch). Spin-manifolds and group actions. In *Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham)*, pages 18–28. Springer, New York.

[57] Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30. Univ. of Tokyo Press, Tokyo.

[58] Topology of elliptic operators. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968)*, pages 101–119. Amer. Math. Soc., Providence, R.I.

[59] *Vector fields on manifolds*. Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, Heft 200. Westdeutscher Verlag, Cologne.

[60] Resolution of singularities and division of distributions. *Comm. Pure Appl. Math.*, 23:145–150.

## 1971

[61] Elliptic operators and singularities of vector fields. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 207–209. Gauthier–Villars, Paris.

[62] On the work of Serge Novikov. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 11–13. Gauthier–Villars, Paris.

[63] (with G.B. Segal). Exponential isomorphisms for $\lambda$-rings. *Quart. J. Math. Oxford Ser. (2)*, 22:371–378.

[64] Riemann surfaces and spin structures. *Ann. Sci. École Norm. Sup. (4)*, 4:47–62.

[65] (with I.M. Singer). The index of elliptic operators. IV. *Ann. of Math. (2)*, 93:119–138. Also available in *Uspehi Mat. Nauk*, 27(4(166)):161–188, 1972 (in Russian).

[66] (with I.M. Singer). The index of elliptic operators. V. *Ann. of Math. (2)*, 93:139–149. Also available in *Uspehi Mat. Nauk*, 27(4(166)):161–188, 1972 (in Russian).

## 1972

[67] (with J.L. Dupont). Vector fields with finite singularities. *Acta Math.*, 128:1–40.

## 1973

[68] (with R. Bott and V.K. Patodi). On the heat equation and the index theorem. *Invent. Math.*, 19:279–330. Errata, *Invent. Math.* 28:277–280, 1975. Also available in *Matematika: Periodicheskii sbornik perevodov inostrannykh statei. Tom* 17, *vyp.* 6, pages 3–48, Izdat. "Mir", Moscow (in Russian).

[69] (with R. Bott and L. Gårding). Lacunas for hyperbolic differential operators with constant coefficients. II. *Acta Math.*, 131:145–206, 1973. Also available in *Uspekhi Mat. Nauk*, 39(3(237)):171–224, 1984 (in Russian).

[70] (with V.K. Patodi and I.M. Singer). Spectral asymmetry and Riemannian geometry. *Bull. London Math. Soc.*, 5:229–234.

[71] *Eigenvalues and Riemannian geometry (notes by Kôji Shiga)*, Tokyo. Iwanami Shoten. Special issue of Sûgaku **25** (1973), no. 4.

[72] *The index of elliptic operators. Distributed in conjunction with the colloquium lectures given at the Fairmont Hotel, Dallas, Texas, January 25–28, 1973 seventy-ninth annual meeting of the American Mathematical Society*. American Mathematical Society (AMS). 19 p., Providence, RI.

## 1974

[73] Invariant theory and Riemannian geometry. In *Proceedings of the International Conference on Prospects in Mathematics (Taniguchi Internat. Sympos., Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1973)*, pages 19–24. Res. Inst. Math. Sci., Kyoto Univ., Kyoto.

[74] *Elliptic operators and compact groups*. Springer-Verlag, Berlin. Lecture Notes in Mathematics, Vol. 401.

[75] (with L. Smith). Compact Lie groups and the stable homotopy of spheres. *Topology*, 13:135–142.

[76] How research is carried out. *Bull. IMA*, 10:232–234.


## 1975


[77] Classical groups and classical differential operators on manifolds. In *Differential operators on manifolds (Centro Internaz. Mat. Estivo (C.I.M.E.), III Ciclo, Varenna, 1975)*, pages 5–48. Cremonese, Rome.

[78] The heat equation in Riemannian geometry (after Patodi, Gilkey, etc.). In *Séminaire Bourbaki, Vol. 1973/1974, 26ème année, Exp. No. 436*, pages 1–11. Lecture Notes in Math., Vol. 431. Springer, Berlin.

[79] (with V.K. Patodi and I.M. Singer). Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69.

[80] (with V.K. Patodi and I.M. Singer). Spectral asymmetry and Riemannian geometry. II. *Math. Proc. Cambridge Philos. Soc.*, 78(3):405–432.

[81] Eigenvalues and Riemannian geometry. In *Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973)*, pages 5–9. Univ. Tokyo Press, Tokyo.


## 1976


[82] Singularities of functions. *Bull. Inst. Math. Appl.*, 12(7):203–206. Papers presented at the Symposium on Excitement in Mathematics (Cambridge, 1975).

[83] Bakerian Lecture, 1975. Global geometry. *Proc. Roy. Soc. London Ser. A*, 347(1650):291–299.

[84] Elliptic operators, discrete groups and von Neumann algebras. In *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, pages 43–72. Astérisque, No. 32–33. Soc. Math. France, Paris.

[85] (with E. Rees). Vector bundles on projective 3-space. *Invent. Math.*, 35:131–153.

[86] (with V.K. Patodi and I.M. Singer). Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.*, 79(1):71–99.


## 1977


[87] R.C. Kirby and L.C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*. Princeton University Press, Princeton, N.J. With notes by J.W. Milnor and M.F. Atiyah, Annals of Mathematics Studies, No. 88.

[88] (with R.S. Ward). Instantons and algebraic geometry. *Comm. Math. Phys.*, 55(2):117–124.

[89] A survey of $K$-theory. In *$K$-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975)*, pages 1–9. Lecture Notes in Math., Vol. 575. Springer, Berlin.

[90] (with W. Schmid). A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math.*, 42:1–62. Erratum, *Invent. Math.*, 54(2):189–192, 1979.

[91] (with N.J. Hitchin and I.M. Singer). Deformations of instantons. *Proc. Nat. Acad. Sci. U.S.A.*, 74(7):2662–2663.

[92] William Vallance Douglas Hodge. *Bull. London Math. Soc.*, 9(1):99–118. (1 plate).

[93] Trends in Pure Mathematics. In *3rd International Congress on Mathematical Education, Karlsruhe 1976*, pages 61–74. 3rd ICME Proc.


## 1978


[94] (with N.J. Hitchin, V.G. Drinfel'd, and Y.I. Manin). Construction of instantons. *Phys. Lett. A*, 65(3):185–187.

[95] (with J.D.S. Jones). Topological aspects of Yang–Mills theory. *Comm. Math. Phys.*, 61(2):97–118.

[96] Geometry and physics. *Sûgaku*, 30(2):128–131. Special issue on the 100th anniversary of the Mathematical Society of Japan.

[97] (with N.J. Hitchin and I.M. Singer). Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461.

[98] Geometry of Yang–Mills fields. In *Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977)*, volume 80 of *Lecture Notes in Phys.*, pages 216–221. Springer, Berlin.

[99] The unity of mathematics. *Bull. London Math. Soc.*, 10(1):69–76. Also available in *Fiz.-Mat. Spis. B"lgar. Akad. Nauk.*, 22(55)(1):11–18, 1979 (in Bulgarian).


## 1979


[100] Lines of development in pure mathematics. *Normat*, (1):10–20, 48. A Norwegian translation from English of the author's original lecture [Third International Congress on Mathematical Education (Karlsruhe, 1976), pp. 61–74, ICME Proc., Karlsruhe, 1977].

[101] Introduction (p. 1). The Harish-Chandra character (pp. 176–181). G.L. Luke, editor. *Representation theory of Lie groups*, volume 34 of *London Mathematical Society Lecture Note Series*, Cambridge. Cambridge University Press.

[102] *Geometry on Yang–Mills fields*. Scuola Normale Superiore Pisa, Pisa.


## 1980


[103] Real and complex geometry in four dimensions. In *The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979)*, pages 1–10. Springer, New York.

[104] (with R. Bott). Yang–Mills and bundles over algebraic curves. In *Geometry and analysis*, pages 11–20. Indian Acad. Sci., Bangalore.

[105] Geometrical aspects of gauge theories. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 881–885, Helsinki. Acad. Sci. Fennica.

[106] Remarks on Morse theory. In *Recent Developments in Gauge Theories*. Plenim Pub. Corp.


## 1981


[107] (with R. Bott). Yang–Mills and bundles over algebraic curves. *Proc. Indian Acad. Sci. Math. Sci.*, 90(1):11–20.

[108] Green's functions for self-dual four-manifolds. In *Mathematical analysis and applications, Part A*, volume 7 of *Adv. in Math. Suppl. Stud.*, pages 129–158. Academic Press, New York.


## 1982


[109] Geometry and monopoles. In *Monopoles in quantum field theory (Trieste, 1981)*, pages 3–20. World Sci. Publishing, Singapore.

[110] What is geometry? *Math. Gaz.*, 66(437):179–184. Also available in *Normat*, 31(2):70–74, 1983 (in Norwegian) and in *Pokroky Mat. Fyz. Astronom.*, 29(4):213–217, 1984 (in Czech).

[111] (with H. Donnelly and I.M. Singer). Geometry and analysis of Shimizu $L$-functions. *Proc. Nat. Acad. Sci. U.S.A.*, 79(18):5751.

[112] Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15.

[113] Gauge theory and algebraic geometry. Differential geometry and differential equations, Proc. 1980 Beijing Sympos., Vol. 1, pages 1–20.

[114] *Solutions of classical equations*. Gauge theories: fundamental interactions and rigorous results, Lect. 1981 Int. Summer Sch. Theor. Phys., Poiana Brasov/Romania, Prog. Phys. 5, pages 207–219.

## 1983

[115] (with A.N. Pressley). Convexity and loop groups. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 33–63. Birkhäuser Boston, Boston, MA.

[116] (with H. Donnelly and I.M. Singer). Eta invariants, signature defects of cusps, and values of $L$-functions. *Ann. of Math. (2)*, 118(1):131–177. Addendum, in *Ann. of Math. (2)*, 119(3):635–637, 1984.

[117] Angular momentum, convex polyhedra and algebraic geometry. *Proc. Edinburgh Math. Soc. (2)*, 26(2):121–133.

[118] (with R. Bott). The Yang–Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615.

## 1984

[119] M. Atiyah, C.A.R. Hoare, and J.C. Shepherdson, editors. *Mathematical logic and programming languages*. Royal Society, London. Philos. Trans. Roy. Soc. London Ser. A **312** (1984), no. 1522.

[120] Anomalies and index theory. In *Supersymmetry and supergravity/nonperturbative QCD (Mahabaleshwar, 1984)*, volume 208 of *Lecture Notes in Phys.*, pages 313–322. Springer, Berlin.

[121] Instantons in two and four dimensions. *Comm. Math. Phys.*, 93(4):437–451.

[122] The moment map in symplectic geometry. In *Global Riemannian geometry (Durham, 1983)*, Ellis Horwood Ser. Math. Appl., pages 43–51. Horwood, Chichester.

[123] The Yang–Mills equations and the structure of 4-manifolds. In *Global Riemannian geometry (Durham, 1983)*, Ellis Horwood Ser. Math. Appl., pages 11–17. Horwood, Chichester.

[124] (with I.M. Singer). Dirac operators coupled to vector potentials. *Proc. Nat. Acad. Sci. U.S.A.*, 81(8, Phys. Sci.):2597–2600.

[125] (with R. Bott). The moment map and equivariant cohomology. *Topology*, 23(1):1–28.

[126] Speech on conferment of Feltrinelli Prize. Accademia Nazionale dei Lincei, pages 183–188.

[127] *Geometry and analysis in the nineteen eighties*. Coll. for the 25th anniversary. IHES.

[128] Mathematics and the computer revolution. *Nuova Civilta delle Macchine*, II(3).

## 1985

[129] Topological aspects of anomalies. In *Symposium on anomalies, geometry, topology (Chicago, Ill., 1985)*, pages 22–32. World Sci. Publishing, Singapore.

[130] (with N.J. Hitchin). Low-energy scattering of nonabelian magnetic monopoles. *Philos. Trans. Roy. Soc. London Ser. A*, 315(1533):459–469.

[131] M. Atiyah, J.D. Gibbon, and G. Wilson, editors. *New developments in the theory and application of solitons*. Philos. Trans. Roy. Soc. London Ser. A **315** (1985), no. 1533. Appears as a book under the same title [ISBN 0-85403-2517].

[132] Circular symmetry and stationary-phase approximation. *Astérisque*, (131):43–59. Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983).

[133] Identifying progress in mathematics. In *The identification of progress in learning (Colmar, 1983)*, pages 24–41. Cambridge Univ. Press, Cambridge.

[134] Eigenvalues of the Dirac operator. In *Workshop Bonn 1984 (Bonn, 1984)*, volume 1111 of *Lecture Notes in Math.*, pages 251–260. Springer, Berlin.

[135] Commentary on the article of Yu.I. Manin: "New dimensions in geometry". In *Workshop Bonn 1984 (Bonn, 1984)*, volume 1111 of *Lecture Notes in Math.*, pages 103–109. Springer, Berlin.

[136] (with N.J. Hitchin). Low energy scattering of nonabelian monopoles. *Phys. Lett. A*, 107(1):21–25.

## 1987

[137] On the work of Simon Donaldson. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 3–6, Providence, RI. Amer. Math. Soc.
[138] The logarithm of the Dedekind $\eta$-function. *Math. Ann.*, 278(1–4):335–380.
[139] Magnetic monopoles in hyperbolic spaces. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 1–33. Tata Inst. Fund. Res., Bombay.

## 1988

[140] Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989).
[141] The impact of physics on geometry. In *Differential geometrical methods in theoretical physics (Como, 1987)*, volume 250 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 1–9. Kluwer Acad. Publ., Dordrecht.
[142] Topology and differential equations. In *ICIAM '87: Proceedings of the First International Conference on Industrial and Applied Mathematics (Paris, 1987)*, pages 45–52, Philadelphia, PA. SIAM.
[143] New invariants of 3- and 4-dimensional manifolds. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 285–299. Amer. Math. Soc., Providence, RI. Also in Uspehi Mat. Nauk 45(4 (274)):3–16, 1990 (in Russian).
[144] *Collected works. Vol. 1*. Early papers: general papers. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.
[145] *Collected works. Vol. 2*. $K$-theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.
[146] *Collected works. Vol. 3*. Index theory: 1. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.
[147] *Collected works. Vol. 4*. Index theory: 2. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.
[148] *Collected works. Vol. 5*. Gauge theories. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York.
[149] (with N.J. Hitchin). *The geometry and dynamics of magnetic monopoles*. M.B. Porter Lectures. Princeton University Press, Princeton, NJ. Also available in Russian, "Mir", Moscow, 1991.
[150] Geometry, Topology and Physics (Delivered as the Eleventh Arthur Milne Lecture (1896–1950) at Oxford University). *Quart. Journ. Astrophysics Soc.*, 29:287–299.

## 1989

[151] (with G.B. Segal). On equivariant Euler characteristics. *J. Geom. Phys.*, 6(4):671–677.
[152] M. Atiyah, J.R. Ellis, M.B. Green, and C.H. Llewellyn-Smith, editors. *Physics and mathematics of strings*. Philos. Trans. Roy. Soc. London Ser. A **329** (1989), no. 1605.
[153] The geometry and physics of knots. In *Miniconference on Geometry and Physics (Canberra, 1989)*, volume 22 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 1–17. Austral. Nat. Univ., Canberra.
[154] The frontier between geometry and physics. *Jahresber. Deutsch. Math.-Verein.*, 91(4):149–158. Also available in *Fiz.-Mat. Spis. B"lgar. Akad. Nauk.*, 33(66)(1–2):41–49, 1991 (in Bulgarian).
[155] W.V.D. Hodge. *The theory and applications of harmonic integrals*. Cambridge Mathematical Library. Cambridge University Press, Cambridge. Reprint of the 1941 original, with a foreword by M. Atiyah.
[156] (with N.S. Manton). Skyrmions from instantons. *Phys. Lett. B*, 222(3–4):438–442.
[157] Topological quantum field theories (René Thom Symposium). *Inst. Hautes Études Sci. Publ. Math.*, 68:175–186.

## 1990


[158] Quantum field theory and low-dimensional geometry. *Progr. Theoret. Phys. Suppl.*, (102):1–13 (1991).

[159] Representations of braid groups. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, volume 151 of *London Math. Soc. Lecture Note Ser.*, pages 115–122. Cambridge Univ. Press, Cambridge. Notes by S.K. Donaldson.

[160] The icosahedron. *Math. Medley*, 18(1):1–12.

[161] The Jones–Witten invariants of knots. *Astérisque*, (189–190):Exp. No. 715, 7–16. Séminaire Bourbaki, Vol. 1989/90.

[162] (with L. Jeffrey). Topological Lagrangians and cohomology. *J. Geom. Phys.*, 7(1):119–136.

[163] *The geometry and physics of knots*. Lezioni Lincee. [Lincei Lectures]. Cambridge University Press, Cambridge. Also available in Russian, "Mir", Moscow, 1995.

[164] Hyper-Kähler manifolds. In *Complex geometry and analysis (Pisa, 1988)*, volume 1422 of *Lecture Notes in Math.*, pages 1–13. Springer, Berlin.

[165] On framings of 3-manifolds. *Topology*, 29(1):1–7.

[166] The Geometry and Physics of Knots. In *Royal Institution Proceedings*, pages 213–225.


## 1991


[167] On the work of Edward Witten. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 31–35, Tokyo. Math. Soc. Japan.

[168] The European Mathematical Society. In *Miscellanea mathematica*, pages 1–5. Springer, Berlin.

[169] Magnetic monopoles and the Yang-Baxter equations. *Internat. J. Modern Phys. A*, 6(16):2761–2774.


## 1992


[170] Mathematics and the physical world. In *In the forest of symbols (Finnish)*, pages 186–210. Art House, Helsinki.

[171] A new knot invariant. I. In *Topological quantum field theories and geometry of loop spaces (Budapest, 1989)*, pages 1–9. World Sci. Publ., River Edge, NJ.

[172] A new knot invariant. II. Topological quantum field theories and the Jones polynomial. In *Topological quantum field theories and geometry of loop spaces (Budapest, 1989)*, pages 10–15. World Sci. Publ., River Edge, NJ.

[173] *The mysteries of space*. Selected Lectures in Mathematics. American Mathematical Society, Providence, RI. The 1991 Josiah Willard Gibbs Lecture presented in San Francisco, California, January 1991.

[174] Address of the president, Sir Michael Atiyah, given at the anniversary meeting on 29 November 1991. *Notes and Records Roy. Soc. London*, 46(1):155–169.


## 1993


[175] Mathematics as a basic science. *Current Sci.*, 65(12):912–917. Also in *Math. Ed.*, 10(2):85–92.

[176] (with N.S. Manton). Geometry and kinematics of two skyrmions. *Comm. Math. Phys.*, 153(2):391–422.

[177] Mathematics: Queen and Servant of the Sciences. *Proc. Am. Philosophical Soc.*, 137(4):527–531.

[178] American Philosophical Society Dinner Address, 30 April 1993. *Proc. Am. Philosophical Soc.*, 137(4):704–707.

## 1994


[179] Contribution to the collected works of Raoul Bott. In *Raoul Bott: Collected Papers, Vol. 2*, Contemp. Mathematicians, pages xxix–xxx. Birkhäuser Boston, Boston, MA.

[180] Responses to: A. Jaffe and F. Quinn, "Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics" [Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 1, 1–2]. *Bull. Amer. Math. Soc. (N.S.)*, 30(2):178–207. Also in *Pokroky Mat. Fyz. Astronom.*, 41(2):73–81, 1996 (in Czech).


## 1995


[181] The index of elliptic operators on compact manifolds. In *Séminaire Bourbaki, Vol. 8*, pages Exp. No. 253, 159–169. Soc. Math. France, Paris.

[182] Hyperbolic differential equations and algebraic geometry (after Petrowsky). In *Séminaire Bourbaki, Vol. 10*, Exp. No. 319, 87–99. Soc. Math. France, Paris.

[183] Reflections on geometry and physics. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 1–6. Int. Press, Cambridge, MA.

[184] (with E. Brézin, A. Connes, J. Fröhlich, D. Gross, A. Jaffe, L. Kadanoff, J. Lebowitz, and D. Ruelle). Round table: physics and mathematics. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pages 691–705. Int. Press, Cambridge, MA.

[185] Quantum physics and the topology of knots. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pages 5–14. Int. Press, Cambridge, MA.

[186] Floer homology. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 105–108. Birkhäuser, Basel.

[187] Reflections on geometry and physics. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 423–428. Int. Press, Cambridge, MA.

[188] Quantum theory and geometry. *J. Math. Phys.*, 36(11):6069–6072.

[189] S.J. Abas and A.S. Salman. *Symmetries of Islamic geometrical patterns*. World Scientific Publishing Co. Inc., River Edge, NJ. With forewords by A. Moustafa and M. Atiyah.

[190] Address of the President, Sir Michael Atiyah, O. M., given at the anniversary meeting on 30 November 1994. *Notes and Records Roy. Soc. London*, 49(1):141–151.

[191] Book Review of *Conversations on Mind, Matter and Mathematics* by Jean-Pierre Changeux and Alain Connes. T. H. E. S.


## 1996


[192] V.K. Patodi. *Collected papers of V.K. Patodi*. World Scientific Publishing Co. Inc., River Edge, NJ. Edited and with a foreword by M.F. Atiyah and M.S. Narasimhan.

[193] S. Hawking and R. Penrose. *The nature of space and time*. The Isaac Newton Institute Series of Lectures. Princeton University Press, Princeton, NJ. With a foreword by M. Atiyah.

[194] Friedrich Hirzebruch—an appreciation. In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, volume 9 of *Israel Math. Conf. Proc.*, pages 1–5, Ramat Gan. Bar-Ilan Univ.


## 1997


[195] M. Atiyah and D. Iagolnitzer, editors. *Fields Medallists' lectures*, volume 5 of *World Scientific Series in 20th Century Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ.

[196] The index of elliptic operators. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 115–127. World Sci. Publ., River Edge, NJ.

[197] The work of Serge Novikov. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 195–197. World Sci. Publ., River Edge, NJ.

[198] The work of Edward Witten. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 514–518. World Sci. Publ., River Edge, NJ.

[199] The work of Simon Donaldson. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 377–380. World Sci. Publ., River Edge, NJ.
[200] An introduction to topological quantum field theories. *Turkish J. Math.*, 21(1):1–7.
[201] Geometry and physics: where are we going? In *Geometry and physics (Aarhus, 1995)*, volume 184 of *Lecture Notes in Pure and Appl. Math.*, pages 1–7. Dekker, New York.

## 1998

[202] Mathematics and the real world. *Quart. Appl. Math.*, 56(4):807–812.
[203] Duality and quantum field theory. In *Topics in symplectic* 4*-manifolds (Irvine, CA, 1996)*, First Int. Press Lect. Ser., I, pages 1–7. Int. Press, Cambridge, MA.
[204] Roger Penrose—a personal appreciation. In *The geometric universe (Oxford, 1996)*, pages 3–7. Oxford Univ. Press, Oxford.
[205] (with M. Jacob, A. Pais, and D.I. Olive). *Paul Dirac*. Cambridge University Press, Cambridge. Edited by P. Goddard.
[206] The Dirac equation and geometry. In *Paul Dirac*, pages 108–124. Cambridge Univ. Press, Cambridge.
[207] Obituary: John Arthur Todd. *Bull. London Math. Soc.*, 30(3):305–316.

## 1999

[208] Physics and geometry: a look at the last twenty years. In *Algebraic geometry: Hirzebruch 70 (Warsaw, 1998)*, volume 241 of *Contemp. Math.*, pages 1–8. Amer. Math. Soc., Providence, RI.
[209] Obituary: Kunihiko Kodaira. *Bull. London Math. Soc.*, 31(4):489–493.

## 2000

[210] The geometry of classical particles. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 1–15. Int. Press, Somerville, MA.
[211] 100 years of mathematics. *Normat*, 48(3):123–126, 144.
[212] Geometry and physics in the 20th century. In *The mathematical sciences after the year 2000 (Beirut, 1999)*, pages 1–9. World Sci. Publ., River Edge, NJ.
[213] V. Arnold, M. Atiyah, P. Lax, and B. Mazur, editors. *Mathematics: frontiers and perspectives*. American Mathematical Society, Providence, RI.
[214] Some personal reminiscences. In *Oxford figures*, pages 257–267. Oxford Univ. Press, Oxford.

## 2001

[215] $K$-theory past and present. In *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, pages 411–417. Berliner Math. Gesellschaft, Berlin.
[216] Mathematics in the 20th century. *Amer. Math. Monthly*, 108(7):654–666. Also in *Math. Today (Southend-on-Sea)*, 37(2):46–53, *Contemporary trends in algebraic geometry and algebraic topology (Tianjin 2000)*, volume 5 of *Nankai Tracts Math.*, pages 1–21. World Sci. Publ., River Edge, NJ, 2002, *Bull. London Math. Soc.*, 34(1):1–15, 2002, *NTM (N.S.)*, 10(1):25–39, 2002, *Wiadom. Mat.*, 39:47–63, 2003 (in Polish) and *Adv. Math. (China)*, 33(1):26–40, 2004 (in Chinese).
[217] Configurations of points. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.*, 359(1784):1375–1387.
[218] (with J. Maldacena and C. Vafa). An M-theory flop as a large $N$ duality. *J. Math. Phys.*, 42(7):3209–3220.
[219] Equivariant cohomology and representations of the symmetric group. *Chinese Ann. Math. Ser. B*, 22(1):23–30.

## 2002

[220] Science and the Military. In R. Spier, editor, *Science and Technology Ethics*. Routledge, London.

[221] Hermann Weyl. *Biographical Memoirs*, 82:3–17. Nat. Acad. Sci. USA.

[222] *The millennium prize problems*. Springer VideoMATH. Springer-Verlag, Berlin. A lecture by M. Atiyah, CMI Millennium Meeting Collection.

[223] On the unreasonable effectiveness of physics in mathematics. In *Highlights of mathematical physics (London, 2000)*, pages 25–38. Amer. Math. Soc., Providence, RI.

[224] (with E. Witten). $M$-theory dynamics on a manifold of $G_2$ holonomy. *Adv. Theor. Math. Phys.*, 6(1):1–106.

[225] (with R. Bielawski). Nahm's equations, configuration spaces and flag manifolds. *Bull. Braz. Math. Soc. (N.S.)*, 33(2):157–176.

[226] (with P. Sutcliffe). The geometry of point particles. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 458(2021):1089–1115.

[227] M. Chisholm. *Such silver currents*. Lutterworth Press, Cambridge. The story of William and Lucy Clifford, 1845–1929, With a foreword by M. Atiyah.

## 2003

[228] (with Paul Sutcliffe). Polyhedra in physics, chemistry and geometry. *Milan J. Math.*, 71:33–58.

[229] (with J. Berndt). Projective planes, Severi varieties and spheres. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, Surv. Differ. Geom., VIII, pages 1–27. Int. Press, Somerville, MA.

[230] M. Atiyah and D. Iagolnitzer, editors. *Fields Medallists' lectures*, volume 9 of *World Scientific Series in 20th Century Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, second edition.

## 2004

[231] (with G.B. Segal). Twisted $K$-theory. *Ukr. Mat. Visn.*, 1(3):287–330. Translation in Ukr. Math. Bull. 1(3):291–334.

[232] *Collected works. Vol. 6*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York.

[233] Lecture by Prof. Michael Atiyah—The geometry behind some string theory dualities. In *UK-Japan Winter School 2004—Geometry and Analysis Towards Quantum Theory*, volume 30 of *Sem. Math. Sci.*, pages 89–108. Keio Univ., Yokohama.

[234] Bakerian lecture, 1975: Global Geometry. *Amer. Math. Monthly*, 111(8):716–723.

[235] (with M. Hopkins). A variant of $K$-theory: $K_{\pm}$. In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 5–17. Cambridge Univ. Press, Cambridge.

[236] The impact of Thom's cobordism theory. *Bull. Amer. Math. Soc. (N.S.)*, 41(3):337–340 (electronic).

## 2005

[237] Mathematics: Art and Science. *Bulletin of the American Mathematical Society*, 43:87–88.

[238] (with P. Sutcliffe). Skyrmions, Instantons, Mass and Curvature. *Physics Letters B*, (605):106–114.

[239] M. Atiyah, A. Bundy, A. Macintyre, and D. MacKenzie, editors. The nature of mathematical proof: Papers of a Discussion Meeting. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 363(1835):2331–2333.

[240] Einstein and geometry. *Current Sci.*, 89(12):2041–2044.

## 2006

[241] Benjamin Franklin and the Edinburgh Enlightenment. *Proceedings American Philosophical Society*, 150(4):591–606.

[242] G.W. Johnson and M.E. Walker. Sir Michael Atiyah's Einstein lecture: "The nature of space". *Notices Amer. Math. Soc.*, 53(6):674–678.

[243] Mathematics: art and science. *Bull. Amer. Math. Soc. (N.S.)*, 43(1):87–88 (electronic).

[244] The interaction between geometry and physics. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 1–15. Birkhäuser Boston, Boston, MA.

[245] (with G. Segal). Twisted $K$-theory and cohomology. In *Inspired by S.S. Chern*, 5–43, Nankai Tracts Math., 11, World Sci. Publ., Hackensack, NJ.

## 2007

[246] Raoul Bott. *Biographical Memoirs of Fellows of the Royal Society*, 53. DOI 10.1098/rsbm.2007.0006.

[247] Jo Rotblat—Man with a Cause. In R. Braum, R. Hinde, D. Krieger, H. Kroto, and S. Milne, editors, *Joseph Rotblat: Visionary for Peace*. Wiley VCH.

[248] Book reviews on Bourbaki: A Secret Society of Mathematicians and The Artist and the Mathematician. *Notices of the American Mathematical Society*, 54(9):1150–1152.

[249] Beyond string theory? In D. Gross, M. Henneaux, and A. Sevrin, editors, *The Quantum Structure of Space and Time, Proceedings of the 23rd Solvay Conference on Physics, Brussels, Belgium 1–3 December, 2005*. World Scientific Press.

[250] *Siamo tutti Matematici (We are all mathematicians)*. Di Renzo Editore.

[251] Einstein and geometry. In *The legacy of Albert Einstein*, 15–23, World Sci. Publ., Hackensack, NJ.

## 2008

[252] Mind, matter, and mathematics. *Berlin–Brandenburg Academy of Sciences, Berichte und Abhandlungen*, 14:151–161.

# List of Publications for Isadore M. Singer


## 1952

[1] Uniformly continuous representations of Lie groups. *Ann. of Math. (2)*, 56:242–247.
[2] (with R.V. Kadison). Some remarks on representations of connected groups. *Proc. Nat. Acad. Sci. U.S.A.*, 38:419–423.

## 1953

[3] (with W. Ambrose). A theorem on holonomy. *Trans. Amer. Math. Soc.*, 75:428–443.

## 1954

[4] (with R. Arens). Function values as boundary integrals. *Proc. Amer. Math. Soc.*, 5:735–745.

## 1955

[5] Report on group representations. In *Report of an international conference on operator theory and group representations, Arden House, Harriman, N.Y., 1955*, Publ. 387, pages 11–26. National Academy of Sciences—National Research Council, Washington, D.C.
[6] (with J. Wermer). Derivations on commutative normed algebras. *Math. Ann.*, 129:260–264.
[7] Automorphisms of finite factors. *Amer. J. Math.*, 77:117–133.

## 1956

[8] (with R. Arens). Generalized analytic functions. *Trans. Amer. Math. Soc.*, 81:379–393.

## 1957

[9] (with R. V. Kadison). Three test problems in operator theory. *Pacific J. Math.*, 7:1101–1106.
[10] (with K. Hoffman). Maximal subalgebras of $C(\Gamma)$. *Amer. J. Math.*, 79:295–305.

## 1958

[11] (with W. Ambrose). On homogeneous Riemannian manifolds. *Duke Math. J.*, 25:647–669.

## 1959

[12] (with R.V. Kadison). Extensions of pure states. *Amer. J. Math.*, 81:383–400.
[13] (with K. Hoffman). Some problems of Gel'fand. *Uspehi Mat. Nauk*, 14(3 (87)):99–114. Also in *Amer. Math. Soc. Transl. (2)*, 27:143–157, 1963 (in English).
[14] The geometric interpretation of a special connection. *Pacific J. Math.*, 9:585–590.

## 1960

[15] Infinitesimally homogeneous spaces. *Comm. Pure Appl. Math.*, 13:685–697.
[16] (with W. Ambrose and R.S. Palais). Sprays. *An. Acad. Brasil. Ci.*, 32:163–178.
[17] (with R.V. Kadison). Triangular operator algebras. Fundamentals and hyperreducible theory. *Amer. J. Math.*, 82:227–259.
[18] (with K. Hoffman). Maximal algebras of continuous functions. *Acta Math.*, 103:217–241.

## 1963

[19] (with M.F. Atiyah). The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69:422–433.
[20] On the Index of Elliptic Operators. In *Outlines of the Joint Soviet–American Symposium on Partial Differential Equations*, pages 1–7. Novosibirsk.

## 1965

[21] (with S. Sternberg). The infinite groups of Lie and Cartan. I. The transitive groups. *J. Analyse Math.*, 15:1–114.

## 1966

[22] (with V. Guillemin). Differential equations and $G$-structures. In *Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965)*, pages 34–36. Nippon Hyoronsha, Tokyo.

## 1967

[23] (with H.P. McKean, Jr.). Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69.
[24] (with J.A. Thorpe). *Lecture notes on elementary topology and geometry*. Scott, Foresman and Co., Glenview, Ill. Reprint of the original version, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1976.

## 1968

[25] Elliptic operators on manifolds. In *Pseudo-Diff. Operators (C.I.M.E., Stresa, 1968)*, pages 333–375. Edizioni Cremonese, Rome.
[26] (with S. Doplicher and T. Regge). A Geometric Model Showing the Independence of Locality and Positivity of the Energy. *Commun. Math. Phys.* 7(1):51–54.
[27] (with M.F. Atiyah). The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530. Also in *Uspehi Mat. Nauk*, 23(5 (143)):99–142 (in Russian).
[28] (with M.F. Atiyah). The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968. Also in *Uspehi Mat. Nauk*, 24(1 (145)):127–182, 1969 (in Russian).

## 1969

[29] (with M.F. Atiyah). Index theory for skew-adjoint Fredholm operators. *Inst. Hautes Études Sci. Publ. Math.*, (37):5–26.

[30] (with J.A. Thorpe). The curvature of 4-dimensional Einstein spaces. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 355–365. Univ. Tokyo Press, Tokyo.

## 1971

[31] Operator theory and periodicity. In *Proceedings of an International Symposium on Operator Theory (Indiana Univ., Bloomington, Ind., 1970)*, volume 20, pages 949–951.

[32] (with L.A. Coburn, R.G. Douglas, and D.G. Schaeffer). $C^*$-algebras of operators on a half-space. II. Index theory. *Inst. Hautes Études Sci. Publ. Math.*, (40):69–79.

[33] Future extensions of index theory and elliptic operators. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 171–185. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J.

[34] Mod 2 index theory. In *Differentialgeometrie im Großen*, p. 105. Bericht einer Tagung des Mathematischen Forschungsinstituts Oberwolfach, 13.–19. Juli 1969 (Wilhelm Klingenberg, ed.). Berichte aus dem Mathematischen Forschungsinstitut Oberwolfach, Heft 4. Bibliographisches Institut, Mannheim.

[35] (with D.B. Ray). $R$-torsion and the Laplacian on Riemannian manifolds. *Advances in Math.*, 7:145–210.

[36] (with M.F. Atiyah). The index of elliptic operators. IV. *Ann. of Math. (2)*, 93:119–138. Also in *Uspehi Mat. Nauk*, 27(4 (166):161–188, 1972 (in Russian).

[37] (with M.F. Atiyah). The index of elliptic operators. V. *Ann. of Math. (2)*, 93:139–149. Also in *Uspehi Mat. Nauk*, 27(4 (166):161–188, 1972 (in Russian).

## 1972

[38] (with L.A. Coburn and R.G. Douglas). An index theorem for Wiener-Hopf operators on the discrete quarter-plane. *J. Differential Geometry*, 6:587–593. Collection of articles dedicated to S.S. Chern and D.C. Spencer on their sixtieth birthdays.

## 1973

[39] (with L.A. Coburn and R.D. Moyer). $C^*$-algebras of almost periodic pseudo-differential operators. *Acta Math.*, 130:279–307.

[40] (with D.B. Ray). Analytic torsion for complex manifolds. *Ann. of Math. (2)*, 98:154–177.

[41] Recent applications of index theory for elliptic operators. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 11–31. Amer. Math. Soc., Providence, R.I.

[42] (with D.B. Ray). Analytic torsion. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 167–181. Amer. Math. Soc., Providence, R.I.

[43] (with M.F. Atiyah and V.K. Patodi). Spectral asymmetry and Riemannian geometry. *Bull. London Math. Soc.*, 5:229–234.

## 1975

[44] (with J.J. Duistermaat). Isomorphismes entre algèbres d'opérateurs pseudo-différentiels. In *Séminaire Goulaouic-Lions-Schwartz 1974–1975: Équations aux dérivées partielles linéaires et non linéaires, Exp. No. 24*, 9 pp. Centre Math., École Polytech., Paris.

[45] Eigenvalues of the Laplacian and invariants of manifolds. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 187–200. Canad. Math. Congress, Montreal, Que.

[46] (with M.F. Atiyah and V.K. Patodi). Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69.

[47] (with M.F. Atiyah and V.K. Patodi). Spectral asymmetry and Riemannian geometry. II. *Math. Proc. Cambridge Philos. Soc.*, 78(3):405–432.

## 1976

[48] *Connections between geometry, topology, and analysis*. American Math. Society Colloquium Lecture, San Antonio, TX.

[49] (with J.J. Duistermaat). Order-preserving isomorphisms between algebras of pseudo-differential operators. *Comm. Pure Appl. Math.*, 29(1):39–47.

[50] (with M.F. Atiyah and V.K. Patodi). Spectral asymmetry and Riemannian geometry. III. *Math. Proc. Cambridge Philos. Soc.*, 79(1):71–99, 1976.

## 1977

[51] Some remarks on operator theory and index theory. In *K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975)*, pages 128–138. Lecture Notes in Math., Vol. 575. Springer, Berlin.

[52] (with M.F. Atiyah and N.J. Hitchin). Deformations of instantons. *Proc. Nat. Acad. Sci. U.S.A.*, 74(7):2662–2663.

[53] B.B. Morrel and I.M. Singer, editors. *K-theory and operator algebras*. Springer-Verlag, Berlin. Lecture Notes in Mathematics, Vol. 575.

## 1978

[54] (with M.F. Atiyah and N.J. Hitchin). Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461.

[55] Some remarks on the Gribov ambiguity. *Comm. Math. Phys.*, 60(1):7–12.

## 1980

[56] H.H. Wu, S. Kobayashi, I.M. Singer, A. Weinstein, and J. Wolf, editors. *The Chern Symposium 1979*, New York. Springer-Verlag.

## 1981

[57] The geometry of the orbit space for nonabelian gauge theories. *Phys. Scripta*, 24(5):817–820.

## 1982

[58] (with M.F. Atiyah and H. Donnelly). Geometry and analysis of Shimizu $L$-functions. *Proc. Nat. Acad. Sci. U.S.A.*, 79(18):5751.

[59] On Yang–Mills fields. In *Nonlinear problems: present and future (Los Alamos, N.M., 1981)*, volume 61 of *North-Holland Math. Stud.*, pages 35–50. North-Holland, Amsterdam.

[60] Gauge Theories, Relativity, and Geometry. In *Five Year Outlook for Science and Technology*. National Academy of Sciences.

[61] Differential Geometry, Fiber Bundles, and Physical Theories. *Physics Today*, 35:41–44.

## 1983

[62] (with M.F. Atiyah and H. Donnelly). Eta invariants, signature defects of cusps, and values of $L$-functions. *Ann. of Math. (2)*, 118(1):131–177.

## 1984

[63] G. 't Hooft, A. Jaffe, H. Lehmann, P.K. Mitter, I.M. Singer, and R. Stora, editors. *Progress in gauge field theory*, volume 115 of *NATO Advanced Science Institutes Series B: Physics*, New York. Plenum Press.

[64] (with O. Alvarez and B. Zumino). Gravitational anomalies and the family's index theorem. *Comm. Math. Phys.*, 96(3):409–417.

[65] (with M.F. Atiyah and H. Donnelly). Signature defects of cusps and values of $L$-functions: the nonsplit case. Addendum to: "Eta invariants, signature defects of cusps, and values of $L$-functions". *Ann. of Math. (2)*, 119(3):635–637.

[66] (with M.F. Atiyah). Dirac operators coupled to vector potentials. *Proc. Nat. Acad. Sci. U.S.A.*, 81(8, Phys. Sci.):2597–2600.

## 1985

[67] Families of Dirac operators with applications to physics. The mathematical heritage of Élie Cartan (Lyon, 1984). *Astérisque*, (Numero Hors Serie):323–340.

[68] (with B. Wong, S.-T. Yau, and S.S.-T. Yau). An estimate of the gap of the first two eigenvalues in the Schrödinger operator. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(2):319–333.

[69] J. Lepowsky, S. Mandelstam, and I.M. Singer, editors. *Vertex operators in mathematics and physics*, volume 3 of *Mathematical Sciences Research Institute Publications*, New York, 1985. Springer-Verlag.

## 1986

[70] (with G. Moore, J. Harris, and P. Nelson). Modular forms and the cosmological constant. *Phys. Lett. B*, 178(2–3):167–173. Erratum, 201(4):579, 1988.

[71] Mathematics: the unifying thread in science. *Notices Amer. Math. Soc.*, 33(5):716–733. A symposium with contributions by P.A. Griffiths, A.M. Cormack, H.A. Hauptman, S. Weinberg and I.M. Singer. Also in *Pokroky Mat. Fyz. Astronom.*, 34(3):129–142, 1989 (in Czech).

## 1987

[72] The $\eta$-invariant and the index. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 239–258. World Sci. Publishing, Singapore.

## 1988

[73] (with R. Seeley). Extending $\overline{\partial}$ to singular Riemann surfaces. *J. Geom. Phys.*, 5(1):121–136.

[74] (with L. Baulieu). Topological Yang–Mills symmetry. *Nuclear Phys. B Proc. Suppl.*, 5B:12–19. Conformal field theories and related topics (Annecy-le-Vieux, 1988).

[75] Some problems in the quantization of gauge theories and string theories. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 199–216. Amer. Math. Soc., Providence, RI.

## 1989

[76] (with T.R. Ramadas and J. Weitsman). Some comments on Chern–Simons gauge theory. *Comm. Math. Phys.*, 126(2):409–420.

[77] (with L. Baulieu). The topological sigma model. *Comm. Math. Phys.*, 125(2):227–237.

## 1990

[78] J. Glimm, J. Impagliazzo, and I. Singer, editors. *The legacy of John von Neumann*, volume 50 of *Proceedings of Symposia in Pure Mathematics*, Providence, RI. American Mathematical Society.

[79] (with O. Alvarez and P. Windey). Quantum mechanics and the geometry of the Weyl character formula. *Nuclear Phys. B*, 337(2):467–486.

## 1991

[80] The new mathematical physics. In *Elementary particles and the universe (Pasadena, CA, 1989)*, pages 187–191. Cambridge Univ. Press, Cambridge.

[81] S. Gindikin and I.M. Singer, editors. *Geometry and physics*. Pitagora Editrice, Bologna. Essays in honour of I.M. Gel'fand, Reprint of J. Geom. Phys. **5** (1988).

[82] (with L. Baulieu). Conformally invariant gauge fixed actions for 2-D topological gravity. *Comm. Math. Phys.*, 135(2):253–265.

## 1992

[83] (with S. Axelrod). Chern-Simons perturbation theory. In *Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991)*, pages 3–45. World Sci. Publ., River Edge, NJ.

[84] For Chern volume. In *Chern—a great geometer of the twentieth century*, pages 88–90. Int. Press, Hong Kong.

[85] (with O. Alvarez and P. Windey). The supersymmetric $\sigma$-model and the geometry of the Weyl–Kac character formula. *Nuclear Phys. B*, 373(3):647–687.

## 1994

[86] R. Bott, M. Hopkins, A. Jaffe, I. Singer, D. Stroock, and S.-T. Yau, editors. *Current developments in mathematics, 1995*. International Press, Cambridge, MA. Papers from the seminar held in Cambridge, MA, April 1995.

[87] (with S. Axelrod). Chern–Simons perturbation theory. II. In *Perspectives in mathematical physics*, Conf. Proc. Lecture Notes Math. Phys., III, pages 17–49. Int. Press, Cambridge, MA. Also in *J. Differential Geom.*, 39(1):173–213.

[88] *The current interface of geometry and elementary particle physics*. AMS-MAA Joint Lecture Series. American Mathematical Society, Providence, RI. A joint AMS-MAA lecture presented in Baltimore, Maryland, January 1992.

## 1995

[89] On the master field in two dimensions. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, volume 131 of *Progr. Math.*, pages 263–281. Birkhäuser Boston, Boston, MA.

[90] Tribute to I.M. Gel'fand for his 80th birthday celebration. In *Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993)*, Progr. Math., pages xix–xxii. Birkhäuser Boston, Boston, MA, 1995.

## 1996

[91] A tribute to Warren Ambrose. *Notices Amer. Math. Soc.*, 43(4):425–427.

## 1997


[92] R. Bott, A. Jaffe, D. Jerison, G. Lusztig, I. Singer, and S.-T. Yau, editors. *Current developments in mathematics, 1996*. International Press, Boston, MA. Papers from the seminar held in Cambridge, MA, 1996.

[93] D. Jerison, I.M. Singer, and D.W. Stroock, editors. *The Legacy of Norbert Wiener: A Centennial Symposium*, volume 60 of *Proceedings of Symposia in Pure Mathematics*, Providence, RI. American Mathematical Society.


## 1998


[94] (with L. Baulieu and H. Kanno). Cohomological Yang–Mills theory in eight dimensions. In *Dualities in gauge and string theories (Seoul/Sokcho, 1997)*, pages 365–373. World Sci. Publ., River Edge, NJ.

[95] (with L. Baulieu and H. Kanno). Special quantum field theories in eight and other dimensions. *Comm. Math. Phys.*, 194(1):149–175.


## 1999


[96] R. Bott, A. Jaffe, D. Jerison, G. Lusztig, I. Singer, and S.-T. Yau, editors. *Current developments in mathematics, 1997*. International Press, Boston, MA. Papers from the conference held in Cambridge, MA, 1997.

[97] B. Mazur, W. Schmid, S.-T. Yau, D. Jerison, I. Singer, and D. Stroock, editors. *Current developments in mathematics, 1998*, Somerville, MA. International Press.

[98] B. Mazur, W. Schmid, S.-T. Yau, D. Jerison, I. Singer, and D. Stroock, editors. *Current developments in mathematics. 1999*, Somerville, MA. International Press.


## 2000


[99] (with V. Mathai). Twisted $k$-Homology theory, twisted Ext-theory. arXiv:hep-th/0012046.


## 2002


[100] (with O. Alvarez). Beyond the elliptic genus. *Nuclear Phys. B*, 633(3):309–344.


## 2005


[101] (with M.J. Hopkins). Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452.

[102] (with V. Mathai and R.B. Melrose). The index of projective families of elliptic operators. *Geom. Topol.*, 9:341–373 (electronic).


## 2006


[103] D.S. Freed, D.R. Morrison, and I. Singer, editors. *Quantum field theory, supersymmetry, and enumerative geometry*, volume 11 of *IAS/Park City Mathematics Series*. American Mathematical Society, Providence, RI. Papers from the Graduate Summer School of the IAS/Park City Mathematics Institute held in Princeton, NJ, 2001.

[104] (with V. Mathai and R.B. Melrose). Fractional analytic index. *J. Differential Geom.*, 74(2):265–292.

[105] P. Etingof, V. Retakh, and I.M. Singer, editors. *The unity of mathematics*, volume 244 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA. In honor of the ninetieth birthday of I.M. Gelfand, Papers from the conference held in Cambridge, MA, August 31–September 4, 2003.


## 2008


[106] (with V. Mathai and R.B. Melrose). Equivariant and fractional index of projective elliptic operators. *J. Diff. Geom.* 78(3):465–473.

# Curriculum Vitae for Sir Michael Francis Atiyah, OM, FRS, FRSE

| | |
|---|---|
| **Born:** | April 22, 1929, London, England |
| **Degrees/education:** | Trinity College, University of Cambridge, BA, 1952<br>Trinity College, University of Cambridge, PhD, 1955 |
| **Positions:** | Research Fellow, Trinity College, University of Cambridge, 1954–58<br>Assistant Lecturer, University of Cambridge, 1957–58<br>Lecturer, University of Cambridge & Fellow of Pembroke College, University of Cambridge, 1958–61<br>Reader, University of Oxford, 1961–63<br>Savilian Professor of Geometry, and Fellow of New College, University of Oxford, 1963–69<br>Professor of Mathematics, Institute for Advanced Study, Princeton, 1969–72<br>Royal Society Research Professor & Fellow of St. Catherine's College, University of Oxford, 1973–90<br>Director, Isaac Newton Institute for Mathematical Sciences, University of Cambridge, 1990–96<br>Master, Trinity College, University of Cambridge, 1990–97<br>Chancellor, University of Leicester, 1995–2005<br>Honorary Professor, Edinburgh University, 1997–<br>Fellow, Trinity College, University of Cambridge, 1997– |
| **Visiting positions:** | Institute for Advanced Study, Princeton, 1955–56, 1959, 1967–68, 1975<br>Harvard University, 1962, 1964<br>University of Chicago, 1968<br>University of California, Berkeley, 1996<br>California Institute of Technology, 2002<br>University of Michigan, 2004 |

| | |
|---|---|
| **Memberships:** | Fellow of the Royal Society 1962<br>American Academy of Arts & Sciences, 1969<br>Royal Swedish Academy of Sciences, 1972<br>Honorary Fellow, Trinity College, University of Cambridge, 1976<br>German Academy of Scientist Leopoldina, 1977<br>Académie des Sciences, France, 1978<br>National Academy of Sciences, USA, 1978<br>Royal Irish Academy, 1979<br>Honorary Fellow, Pembroke College, University of Cambridge, 1983<br>Third World Academy of Science (Associate Founding Fellow), 1983<br>Honorary Member, Royal Institution, 1991<br>Honorary Fellow, St. Catherine's College, University of Oxford, 1991<br>Australian Academy of Sciences, 1992<br>Honorary Fellow, Darwin College, University of Cambridge, 1992<br>Ukrainian Academy of Sciences, 1992<br>Honorary Fellow, Royal Academy of Engineering, 1993<br>Honorary Professor, Chinese Academy of Sciences<br>Indian National Science Academy, 1993<br>Russian Academy of Sciences, 1994<br>Georgian Academy of Sciences, 1996<br>Academy of Physical, Mathematical and Natural Sciences of Venezuela, 1997<br>American Philosophical Society, 1998<br>Accademia Nazionale dei Lincei, Rome, 1999<br>Honorary Fellow, New College, University of Oxford, 1999<br>Honorary Fellow, Faculty of Actuaries, 1999<br>Honorary Fellow, Academy of Medical Sciences, 2000<br>Royal Norwegian Society of Sciences and Letters, 2001<br>Czechoslovakia Union of Mathematics<br>Moscow Mathematical Society<br>Spanish Royal Academy of Sciences, 2002<br>Lebanese Academy of Sciences, 2008<br>Norwegian Academy of Science and Letters, 2009 |
| **Awards and prizes:** | Fields Medal, 1966<br>Royal Medal, Royal Society, 1968<br>De Morgan Medal, 1980<br>Feltrinelli Prize, 1981<br>Knight Bachelor, 1983<br>King Faisal International Prize for Science, 1987<br>Copley Medal, Royal Society, 1988 |

Gunning Victoria Jubilee Prize, Royal Society of Edinburgh, 1990
Order of Merit, 1992
Benjamin Franklin Medal, 1993
Jawaharlal Nehru Memorial Medal, 1993
Commander of the Order of Cedars, Lebanon, 1994
Freedom of the City of London, 1996
Order of Andres Bello (1st Class), Republic of Venezuela, 1997
Royal Medal, Royal Society of Edinburgh, 2003
Abel Prize, 2004
Order of Merit (Gold), Lebanon, 2005
President's Medal, Institute of Physics, 2008

**Honorary degrees:**
University of Bonn, 1968
University of Warwick, 1969
University of Durham, 1979
University of St. Andrews, 1981
Trinity College, Dublin, 1983
University of Chicago, 1983
University of Cambridge, 1984
University of Edinburgh, 1984
University of Essex, 1985
University of London, 1985
University of Sussex, 1986
University of Ghent. 1987
University of Reading, 1990
University of Helsinki, 1990
University of Leicester, 1991
Rutgers University, 1992
University of Salamanca, 1992
University of Montreal, 1993
University of Waterloo, 1993
University of Wales, 1993
Lebanese University, 1994
Queen's University, Kingston, Canada, 1994
University of Keele, 1994
University of Birmingham, 1994
Open University, 1995
University of Manchester, 1996
Chinese University of Hong Kong, 1996
Brown University, 1997
University of Oxford, 1998
University of Wales, Swansea, 1998
Charles University, Prague, 1998
Heriot–Watt University, 1999
University of Mexico, 2001

American University of Beirut, 2004
University of York, 2005
Harvard University, 2006
Scuola Normale, Pisa, 2007
Universitat Politècnica de Catalunya, 2008

**Presidencies:** London Mathematical Society, 1974–76
Mathematical Association, 1981–82
European Mathematical Council, 1978–90
Royal Society, 1990–95
Pugwash Conference on Science & World Affairs, 1997–2002
CAMS, American University of Beirut, International Advisory Committee, 1999–
Royal Society of Edinburgh, 2005–

# Curriculum Vitae for Isadore Manual Singer

| | |
|---|---|
| **Born:** | May 3, 1924 Detroit, USA |
| **Degrees/education:** | B.S. University of Michigan, 1944<br>M.S. University of Chicago, 1948<br>Ph.D. University of Chicago, 1950 |
| **Positions:** | C.L.E. Moore Instructor, MIT, 1950–52<br>Assistant Professor, University of California, Los Angeles, 1952–54<br>Professor, MIT, 1956–70<br>Norbert Wiener Professor, MIT, 1970–79<br>Professor, University of California, Berkeley, 1979–83<br>Miller Professor, University of California, Berkeley, 1972–83<br>John D. MacArthur Professor of Mathematics, MIT, 1983–87<br>Institute Professor, MIT, 1987– |
| **Visiting positions:** | Assistant Professor, Columbia University, 1955<br>Member, Institute for Advanced Study, Princeton, 1956<br>Professor, University of California, Berkeley, 1977–79 |
| **Memberships:** | American Academy of Arts and Sciences, 1959<br>National Academy of Sciences, 1968<br>American Philosophical Society, 1983<br>Norwegian Academy of Science and Letters, 2009 |
| **Awards and prizes:** | Bôcher Memorial Prize, 1969<br>National Medal of Science, 1983<br>Eugene Wigner Medal, 1988<br>Chair of Geometry and Physics, Foundations of France, 1988–89<br>AMS Award for Distinguished Public Service, 1993<br>Steele Prize for Lifetime Achievement, 2000<br>Abel Prize, 2004<br>James Rhyne Killian Faculty Achievement Award (MIT), 2005 |

| | |
|---|---|
| **Honorary degrees:** | Tulane University, 1981<br>University of Michigan, 1989<br>University of Illinois at Chicago, 1990<br>University of Chicago, 1993<br>University of Miami, 2002 |
| **Presidencies:** | Vice-President AMS 1970–72 |