THE ABEL PRIZE 2022

A Glimpse of the Laureate's Work

By Alex Bellos

The Abel Committee has awarded Dennis Parnell Sullivan the 2022 Abel Prize for his contributions in the fields of topology and dynamical systems. The texts below give a brief explanation of some of his work in both of these areas.

Topology

In the 19th century, mathematicians started to look more closely at the essential properties of shapes. A new field of geometry emerged in which two objects are considered to be the same if one of them can be turned into the other by stretching or squeezing, but without any tearing or gluing. In this field, called topology, the letter 'a' is the same as the letter 'b', a square is the same as a circle, and a teacup is the same as a donut.

A basic concept in topology is the "manifold", a shape that is the same everywhere, meaning that it has no end points, edge points, crossing points or branching points. The classification of manifolds – that is, how many different types of manifolds there are and what they are like – has been one of the fundamental areas of topological research since the subject's inception. This is the area in which Dennis Sullivan began his career, the subject of his thesis and important early work.

Let's begin our own simplified classification, starting with manifolds in one dimension. Shapes in one dimension are perhaps most easily thought of as shapes made from string. We can make the letter 'a' from string, but clearly it is not a manifold because it has two end points, at the tip and base of the letter. Neither are the letters 'b' or 'c'. However, the letter 'o' made from string *is* a manifold: it has no end points, crossing points or branches. In fact, the 'o', a closed loop, is the only one-dimensional manifold that can be made out of a finite amount string.

Now, we move up to two dimensions. Shapes in two dimensions are perhaps most easily thought of as shapes made from sheets. A piece of paper is a two-dimensional sheet (if we ignore thickness), but it is not a manifold because it has an edge. The sphere (mathematically speaking, the sphere is the surface of a ball), however *is* a manifold. Wherever you are on a sphere your immediate surroundings look exactly the same.

The torus, the surface of a donut shape, is also a manifold. The double torus, which looks like the surface of a figure-of-eight pretzel, is also a manifold. In fact, the triple torus, the quadruple torus, and so on, are all manifolds. Summarising our two-dimensional classification: the sphere, and family of tori, are the only two-dimensional, orientable manifolds that can be made with a finite amount of sheet material.

Onwards to three dimensions. Shapes in three dimensions are perhaps most easily thought of as shapes made out of dough. But here our visual analogies break down, and we head into abstraction. Notice how one-dimensional, string manifolds like the letter 'o' exist in in two dimensions, and how the two-dimensional torus exists in three. Likewise, three-dimensional dough manifolds exist in four dimensions or more. These shapes cannot be constructed in the three-dimensional space we live in.

The classification of dough manifolds is the subject of the Poincaré conjecture, which was one of the most famous open problems in all mathematics, until the Russian mathematician Grigori Perelman proved it in 2002 and 2003. (Perelman won $1m for his proof, and surprised many people by turning it down.)

We continue up the dimensions. The classification of four-dimensional manifolds is full of open problems and mysteries.

Yet curiously, classification gets easier once we reach manifolds of dimension five and above. Topologists use "surgery theory" to operate on these manifolds and construct new ones. In lay terms, the more dimensions there are, the more "room" there is to move around.

Dennis Sullivan's thesis and early work was on surgery theory. He helped figure out what sort of things you could feed into the surgery program. One of his innovations was to organise surgery theory using "classifying spaces", and to use these spaces as a key to understanding all high dimensional manifolds. His work has helped provide a full picture of what manifolds there are in five and more dimensions, and how they behave.

Dynamical systems

In the mid 1970s, computers stimulated much new mathematical research. It became possible, for example, to investigate the behaviour of systems that relied on many repeated calculations, some of which revealed fascinating and beautiful fractal shapes.

Mathematical biologists, for example, devised models to show how animal populations rise and fall. This simple formula, called the logistic map, captures how an animal population changes from year to year.

$$x_{n+1} = rx_n (1 - x_n)$$

The value x_n is a number between 0 and 1, and represents the size of an animal population at year n as a proportion of the maximum population. The value x_{n+1} is the size of the population at year $n + 1$. And the parameter r is the reproductive rate, or fertility, of the system.

The logistic map is iterative, meaning that, starting with a population size in year 1, you calculate the population in year 2, then plug that value back into the equation to get year 3, then year 4, and so on. The equation captures both how populations grow proportionately (that's the rx_n part), and how they fall as overpopulation puts stress on limited resources (that's the $1 - x_n$ part).

The logistic map reveals unexpectedly complex behaviour depending on the value of r, as illustrated in the graph below, which plots the value of r along the horizontal axis and the limit value of the population along the vertical axis.

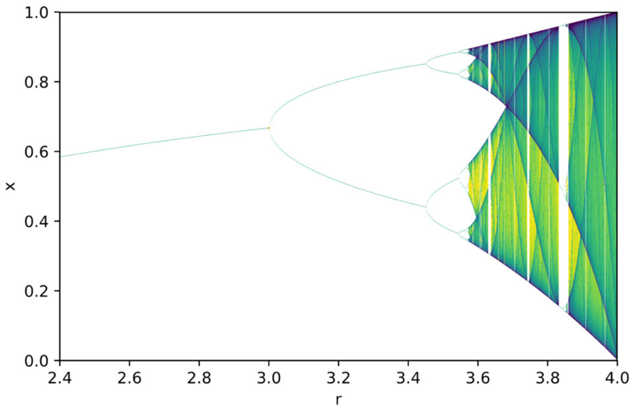

Image credit: Morn, *Wikipedia*, https://en.wikipedia.org/wiki/Bifurcation_diagram#/media/File:Logistic_Map_Bifurcation_Diagram,_Matplotlib.svg

For example, when r is between 2.4 and 3, the population will eventually settle at a fixed value, whatever its initial size, hence the single line on the graph.

When r reaches 3, however, the line forks, meaning that the population does not eventually settle at a single value. Instead, the limit population oscillates in alternate years between two values. As r increases both of these branches fork again, at which point the population oscillates between four values.

The graph, also known as a bifurcation diagram, is one of the most famous mathematical images from the 1970s. The cascades of period doublings are an example of what popularly became known as chaos theory, in which tiny changes in initial conditions can have hugely different consequences. Another lay term for this phenomenon is the “butterfly effect”.

The physicist Mitchell Feigenbaum discovered a fascinating feature of the logistic map: the ratios of the distances between the bifurcation points converge to a fixed number, 4.6692…, called the Feigenbaum constant. In fact, the Feigenbaum constant appears not only with the iterative formula $rx_n (1 - x_n)$, as above, but also with other formulae. It is a universal feature of this kind of system, independent of the fine details of the formula.

Dennis Sullivan showed that the limits of cascades of period doublings are universal. His work in this area led to a deeper understanding of the concept of “renormalization” which now forms part of the foundations of the field. His novel approach revealed how the rich theory of complex numbers can be leveraged to understand the emergence of rigidity phenomenon in real dynamics.

Topology and dynamical systems inhabit different mathematical landscapes. Yet Sullivan’s work can be seen as part of a single and consistent visionary endeavour, the study of geometric structures on spaces, whether that space is a manifold or a fractal. Sullivan’s wide interests and deep insights have made him, in the words of his Abel Prize citation: “a true virtuoso.”