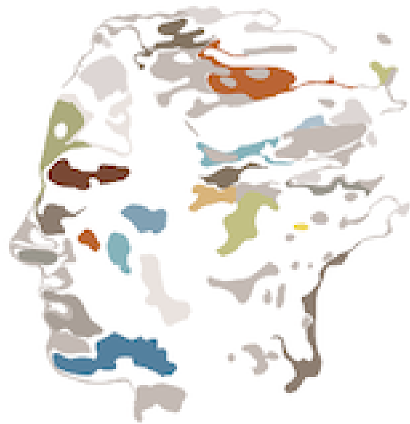


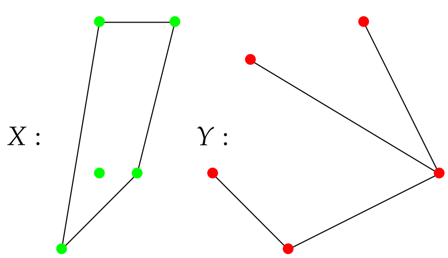
Dennis Parnell Sullivan
Abel Prize Laureate 2022

An algebraic model for topological spaces

The Abel Committee states in the citation:

... Sullivan´s model is based on differential forms, an idea of multivariable calculus, enabling direct connection to geometry and analysis. This made a major part of algebraic topology suitable for calculation, and has proved revolutionary. ...

Algebraization of topology



The two figures X and Y both have 5 vertices and 5 edges. But even if the two pairs of numbers are the same, this does not imply that the figures are the same. Admittedly, it is not that hard to describe the difference between the two figures, but it was still a huge step forward for topology when Emmy Noether in the mid-1920s introduced the idea of introducing algebras into the topology. Her suggestion was to replace the number of certain objects with the corresponding number of copies of $\mathbb{Z}$ and leave the relationship between edges and the corners are encoded in an function d between the two versions of 5 copies of $\mathbb{Z}$:

$$\mathbb{Z}^5 \xrightarrow{d} \mathbb{Z}^5$$

This chain complex of the topological space can be effectively used to calculate some topological invariants of the two figures; For X we get the homology groups

$$H_0(X,\mathbb{Z}) = \mathbb{Z}^2, \quad H_1(X,\mathbb{Z}) = \mathbb{Z}$$

which express that X has two components and a loop, while we for Y get

$$H_0(Y,\mathbb{Z}) = \mathbb{Z}, \quad H_1(Y,\mathbb{Z}) = 0$$

Thus Y has one component and no loops. It is implicit in the last two statements that H_0 measures the number of components and that H_1 "counts" the number of loops.

De Rham´s Theorem

Another example of algebraization of topology dates back to the early 1930s and the Swiss mathematician George de Rham. De Rham´s theorem states that for a smooth manifold M without boundary there is an isomorphism between cohomology groups

$$H^n_{dR}(M) \simeq H^n(M,\mathbb{R})$$

for all $n \geq 0$. De Rham cohomology $H^n_{dR}(M)$ is given as the quotient of closed n-forms (forms ω satisfying $d\omega = 0$) by exact forms (written as $\omega = d\eta$) of the manifold M, while $H^n(M,\mathbb{R})$ is closely related to the homology groups of X and Y as given above. The isomorphism between the two cohomology theories is defined such that for any homology class $[c]$ of M an n-form ω is mapped to the integral of the form along the homology class:

$$\omega \mapsto \int_{[c]} \omega$$

Due to Stoke´s theorem the map is well-defined;

$$d\omega \mapsto \int_{[c]} d\omega = \int_{\partial[c]} \omega = 0$$

The de Rham theorem links two different descriptions of the manifold M. An isomorphism in cohomology is often referred to as a quasi-isomorphism of the underlying complexes:

$$(\Omega^\bullet(M), d_{dR}) \xrightarrow{\simeq} (C^\bullet(M), \partial)$$

The Rham complex is built on differential forms and encodes properties of the differentiable structure of M. The complex $(C^\bullet(M), \partial)$ reflects a structure of M as put together by simpler geometric objects, e.g. line segments, triangles, tetrahedra, etc. In cohomology, the geometry of each constituent will be insignificant, what is essential is the combinatorics in the structure. Consider an ordinary circle. If we join together two line segments, we get something that is topologically the same as one line segment. On the other hand, we pairwise join together the endpoints, we get something more like a circle. Cohomology does not care about what the line segments look like, but it captures if we have one or two joints between the segments.

The Sullivan model

The de Rham complex is an example of what is called a commutative differential graded algebra (CDGA). CDGAs are objects with a rich structure and well suited for calculations. The de Rham theorem can be seen as a way of linking the geometric structure as described above to some mathematical object suitable for calculations. The assumption of the de Rham theorem is that the geometric object is a smooth manifold. Sullivan had an ambition of extending the de Rham theorem to include all topological spaces X without the necessary differential structure. Sullivan´s model is his answer to this challenge. In a systematic and step-by-step manner, Sullivan builds a CDGA based on the geometric structure of the space X. The algebra picks out the most important properties of the geometry such that knowledge of the algebra is actually enough to reconstruct the object. Of course not in details, but with emphasis of the geometric structure.

The power of this construction lies in the fact that a geometric structure is replaced by an almost equivalent. algebraic structure with more flexibility when it comes to calculations.

As an example of a Sullivan model, we consider the circle S^1. There are no maps from S^n, $n \geq 2$ into the circle that can not be pulled together to a point. It means that the Sullivan model has only one basis element a in degree 1 and trivial differential. The model is assumed to be graded commutative thus for $a_1 = a_2 = a$ we have

$$a^2 = a_1 \cdot a_2 = (-1)^{1\cdot 1} a_2 \cdot a_1 = -a^2$$

consequently we have $a^2 = 0$.

For the sphere S^2 the situation is a little bit different. For all n-spheres there is a generator a in degree $n = 2$. Contrary to the odd case $n = 1$ in the even case $n = 2$ we have

$$a^2 = a_1 \cdot a_2 = (-1)^{2\cdot 2} a_2 \cdot a_1 = a^2$$

Thus there are no conditions on a^2. To avoid that a^2 contributes to the model we introduce an element b of degree 3, such that$db = a^2$. Thus the Sullivan model for an even sphere becomes

$$(\wedge(a,b), a^2 = db, da = 0)$$

where deg$(a) = 2$ and deg$(b) = 3$.