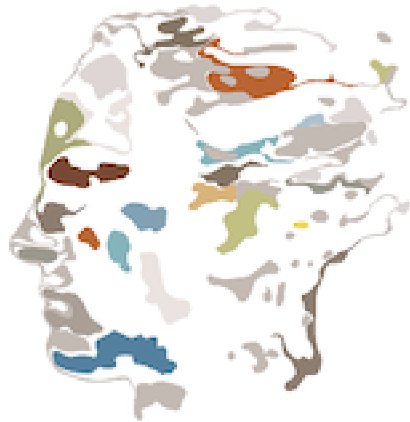


Dennis Parnell Sullivan
Abel Prize Laureate 2022

.. for his groundbreaking contributions to topology in its broadest sense ..

The Abel Committee states in the citation:

"In the late 19th century, a new qualitative way of looking at geometry was born: the subject of topology. In topology a circle and a square are the same, but the surface of the earth and that of a donut are different. Developing a precise language and quantitative tools for measuring the properties of objects that don't change when they are deformed has been invaluable throughout mathematics and beyond, with significant applications in physics to economics to data science.

Dennis Sullivan has repeatedly changed the landscape of topology by defining new concepts, proving landmark theorems answering old conjectures and formulating new problems that have driven the field. He moved from area to area, seemingly effortlessly, using algebraic, analytic, and geometric ideas as a true virtuoso."

Topology

The classical core areas of mathematics are arithmetic and geometry. Arithmetic deals with numbers, while geometry is about studying and classifying shapes, abstract as well as those we observe around us. We have a well-developed ability to distinguish one geometric shapes from another. But we are also good at seeing common features in otherwise different forms. An example is similarity. Similarity means that two figures have the same shape, but different sizes.

In some contexts, it may be appropriate to place objects with even greater variation in the same class: There are many different roundabouts, but we rarely doubt that something is a roundabout. It's not about details in the design or size; it's just about the spot in the middle of the roundabout which you are supposed to round in the right directiorn. The definition of a roundabout has some sort of topological flavour. An open space in the city streets becomes a roundabout only when you remove a centrally located area in the space.

Photo: Adrian Pingstone

In the citation the Abel Committee states that seen through topological glasses a circle and a square are the same, while a ball and a torus are different. Likewise, a circle and a straight line are topologically different, not because one is straight and the other is curved, but because one has two loose ends, while the other is a closed curve and therefore has no loose ends.

Topological classification is a very coarse classification of geometric objects. It is blind to all insignificant details, and focuses on the main features of the geometry. Just like with the roundabouts, it does not matter what they look like, as long as they have the spot in the middle where no one is supposed to drive. Sullivan has been an important contributor to building a toolbox suitable for characterization of topological spaces.

The first tool to be placed in this box, albeit long before Sullivan's time, is the fundamental group.

The fundamental group

Only when humans were able to launch a spacecraft into the space, one could really see with one's own eyes what we had known for hundreds of years: that the earth is round. On a daily basis, the earth appears to be reasonably flat. This reflects the difference between the local and the global. We are tempted to state that topology does not care particularly about the local, but is more concerned with global aspects. Whether a body is rounded at the edges or has sharp corners, it does not matter. But a donut and a ball are basically different. However, a coffee cup with a handle is the same as a donut.

We'll look at the topologist´s way of quantifying the difference between a ball and a donut. More generally, consider a geometric object X, a surface, a body, or something that may be composed of such. A closed curve on the object is a map

$$\gamma : S^1 \to X$$

of a circle into the object. If X is the surface of a ball, we consider this curve as a loop placed on the ball. There are, of course, an infinite number of such loops on a ball and to distinguish topological significants from topological insignificance, we say that two loops are equivalent if we can drag one into the other in a continuous manner. Note that a single point is also formally a loop, since the map $\gamma : S^1 \to X$ can easily map the whole circle into one point. We call this loop the 0-loop. A more general loop will then be equivalent to the 0-loop if we can pinch together the loop to a point, without leaving the surface, nor cutting the loop.

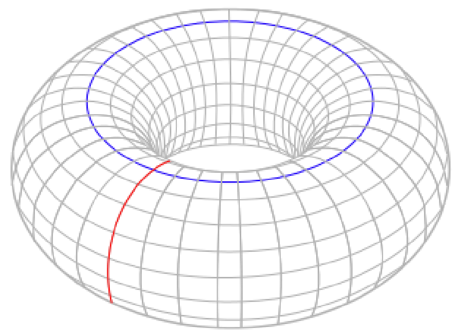
It is easy to see that this is possible as long as the loop is on a ball. On a donut surface (torus) the situation is quite different. If the loop either goes around the hole of the torus, as indicated by the blue curve on the figure, or walk around the tube (follow the red curve), then it will be impossible to pinch the curve to a point. This construction describes what is called the fundamental group π_1 of the geometric object.

$$\pi_1(X) = \{\gamma : S^1 \to X\}/\sim$$

The notation $/\sim$ means that we rather than to study individual curves, we are more concerned with equivalence classes of curves. On a ball S^2 all loops can be pinched into a point and $\pi_1(S^2) = 0$, while on the torus we have two circles, the blue b around the hole, and the red r around the tube. In addition, the loop can run through each trajectory many times in many orders. An example of this element of the fundamental group of the torus is

$$bbrb^{-1}rrb$$

which corresponds to a loop that (read from right to left) first goes around the blue, then twice around the red, then the opposite way around the blue (b^{-1}), around the red and finally twice around the blue.



The fundamental group measures how many equivalence classes of circles are present in the geometric object. We will consider an example where this is an important issue.

Let the geometric object be the electricity network in a region, a little schematic and simplified in the figure below. A significant security factor associated with a power supply network is the robustness of the network in the event of a line break. If we cut the power in one place, there should always be another way around. It means that there must be at least one loop in the network, i.e. an image of the circle S^1, and that the fundamental group of the network contributes to the robustness of the network. The more loops there are, and thus the more complex fundamental group, the easier it is for the electricity to find an alternative route.