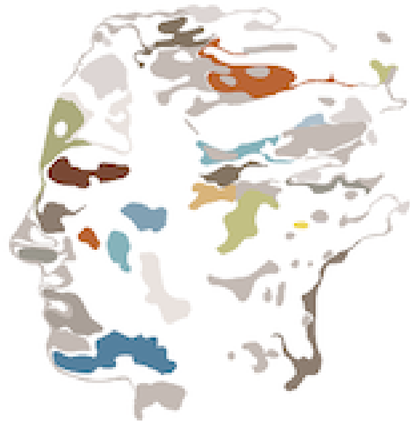

Dennis Parnell Sullivan
Abel Prize Laureate 2022

No-wandering-domains

The Abel Committee states in the citation:

... In dynamics, Sullivan introduced a dictionary between Kleinian groups and iterated rational maps, pivoting on the theory of measurable complex structures. He proved that rational maps have no wandering domains, solving a 60 years old conjecture of Fatou ...

Orbits of dynamical systems

A dynamical system is a mathematical model that describes the time evolution of a physical system. A dynamical system has two main components:

- A set of states for the system, each state being a specification of the values of all the parameters included in the model
- A rule that describes the dynamics of the system, i.e. how the system moves from one state to the next

An orbit of a dynamical system is an ordered set of states where the next state is the one that the law of dynamics assigns to the previous one.

Weather prediction is a dynamical system. A state of the weather system is a list of meteorological data, such as temperature, air pressure, humidity, wind-speed and -direction, and possibly other parameters. Based on one specific state, meteorologists can, by relying on the laws of nature, make a qualified prediction of what the same parameters will look like a moment later. Powerful computers can repeat this process thousands of times and on that basis come up with relatively accurate weather forecasts.

Weather prediction is a complicated dynamical system. To illustrate concepts related to the theory of dynamical systems, we consider a less complex system. States are real numbers and the dynamics is given by the function $f(x) = x^2 - 1$. An orbit of the dynamical system is completely determined by an initial value x_0 and iterations of the function f, i.e. $x_0, f(x_0), f^2(x_0), \dots$. For an initial point $x_0 = 2$, iteration of the function $f(x) = x^2 - 1$ will set up the path

$$\{2, 3, 8, 63, 3968, \dots\}$$

which implies that $f^n(x_0) \to \infty$ when $n \to \infty$. If, on the other hand, we choose the initial point $x_0 = 1$, the orbit will be eventually periodic

$$\{1, 0, -1, 0, -1, \dots\}$$

since $f(-1) = 0$ and $f(0) = -1$. If we change the initial point, to $x_0 = 0.9$ the orbit becomes

$$\{0.9, -0.19, -0.96, -0.07, -0.99, \dots\}$$

This orbit will eventually converge to the periodic orbit $\{-1, 0\}$.

Since the function is the same for all orbits, the orbits must be uniquely determined by their initial point. A classification of the orbits is thus the same as a classification of the initial points.

Different initial points can generate very different orbits. We have seen that the initial point $x_0 = 2$ will force the iteration to go to ∞, in contrast to the initial point $x_0 = 0.9$ where the orbit converges towards a periodic orbit. The latter will in fact be the case for all initial points in the interval

$$\frac{-1-\sqrt{5}}{2} < x_0 < \frac{1+\sqrt{5}}{2}$$

except for $z = \frac{1-\sqrt{5}}{2}$ where the function has a fixed point $f(z) = z$.

We split the set of initial points into two categories, tame and wild. The tame points are characterized by the fact that the orbits of nearby points share many of their essential properties. Points in the interval $\frac{-1-\sqrt{5}}{2} < x_0 < \frac{1+\sqrt{5}}{2}$, where all orbits, except one, converges towards the periodic orbit $\{-1, 0\}$, share this property. We call the set of tame points the Fatou set of the dynamical system. The complement of the Fatou set, consisting solely of the wild points, is called the Julia set. An example of a point in the Julia set for the iteration $f(x) = x^2 - 1$ is the fixed point $z = \frac{1-\sqrt{5}}{2}$. The orbit of z consists of z only, but if we start the iteration away from z, but still close, the iterations will keep on moving away from z and eventually converge towards the periodic orbit $\{-1, 0\}$. An alternative description is that an initial point z of the Julia set is unstable with respect to the type of orbit it generates.

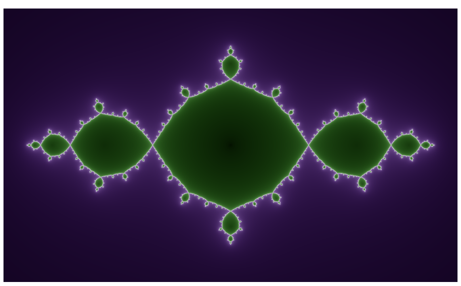

Photo: Georg-Johann Lay

Figure 1: Illustration of the dynamical system given by $f(z) = z^2 - 1$, $z \in \mathbb{C} \cup \{\infty\}$. The white curve is the Julia set, while the purple and the green sets are the two components of the Fatou set. The purple shows all initial points where the iterations will go towards ∞, while initial points in the green set form orbits that eventually converges towards the periodic orbit $\{-1, 0\}$.

The Fatou no-wandering-domain Conjecture

In the 1920s, Pierre Fatou put forward his conjecture about the tame points of a rational dynamical system over the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Sullivan proved the conjecture in 1985 and thereby changed the status of the result from Fatou's conjecture to Sullivan's theorem.

Theorem (Sullivan, 1985). If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of degree $d \geq 2$, then every component U of the Fatou set $F(f)$ is eventually periodic.

The term no-wandering-domain refers to the fact that periodic components are not wandering; a wandering component U satisfies $U \cap f^p(U) = \emptyset$ for all $p \geq 2$. An example of a wandering component is found for the iteration given by $f(z) = z + 2\pi \sin z$. The white fields in the figure illustrates a wandering component of the Fatou set; the orbit of the point $z_0 = \frac{\pi}{2}$ is given by

$$\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$$

Notice that the function $f(z) = z + 2\pi \sin z$ is not rational and thus does not violate Sullivan's theorem.

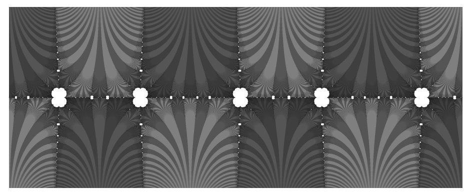

Photo: Lasse Rempe-Gillen

Figure 2: Illustration of the dynamical system given by $f(z) = z + 2\pi \sin z$, $z \in \mathbb{C} \cup \{\infty\}$.

Sullivan's proof of the no-wandering-domain theorem builds on deep insight into the geometry of the extended complex plane and functions defined on it. An assumption that there exists wandering components for a rational function of degree $d \geq 2$ leads to the existence of infinitely many linearly independent rational functions of degree $d \geq 2$. However, it is well known that the vector space of rational functions of degree $d \geq 2$ are finite dimensional, and the contradiction gives us the necessary evidence that the theorem is true.