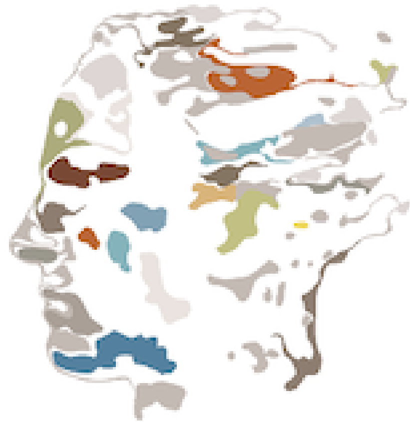

Luis Caffarelli
Abel Prize Laureate 2023

Free Boundary Problems

As on a signal the crowd silences. The two top world tennis players are looking nervous at each other, waiting for the decisive serve in the final match of this year‘s Wimbledon Championships. The top ranked player pulls her sun visor a small inch up and starts bouncing the ball. The ball hits the well-trimmed lawn and with an almost inaudible sound it returns back to the players hand...

Source: Screen dump of: ScienceLuxembourg <https://youtu.be/1yT0hxplVBg>

How can it be that the ball bounces back? The picture of the deformed ball illustrates what happens. Hitting the ground at some speed the ball is pressed together. Because of the rather stable shape of the ball the internal forces of the ball will start pushing the ball back to the original shape. The reforming process of the shape takes place at a certain speed, caused by the elasticity of the ball. This speed is high enough, not only to reshape the ball, but also to give the ball some upwards speed. The experienced tennis player knows that some kinetic energy is lost in the inelastic collision with the ground and adjust the bounce effort in a reasonable way.

For a physicist this explanation of what happens is satisfactory, but the mathematician needs to go deeper. Is it possible to establish a mathematical model for the shape of the ball during the collision? The model should combine the laws of motion, forces caused by the increased pressure in the ball when pressed together and the elasticity of the material of the ball.

Before hitting the ground, we assume that the ball is a perfect sphere. As it hits the flat ground, the contact surface between the ball and the ground will be flat, too. The rest of the ball will look like a part of an oval. A major problem in modelling the deformation is that the contact surface between the ball and the ground will vary during the collision, ranging from one initial point to a certain part of the surface of the ball. Taking into account all the physical laws known to be involved, it is still not an easy task to set up a mathematical model for the deformation of the ball. To find a solution of the model, i.e. to give an nice description of the movements of every spot of the ball, is even harder.

A major problem in finding the solution is that this is a so-called “free boundary problem”, one of this year‘s Abel Laureate Luis Caffarelli‘s specialties. In general, when looking for a solution of a mathematical model, it is of crucial importance to know the geometry of the object in study. In particular you must be able to describe the boundary of the object.

Suppose you put a silver spoon into your tea cup. The heat will spread from the hot tea, through the metal, ending up in your fingertips. To calculate the time-dependent distribution of the heat in the spoon, based on the heat diffusion model of Fourier, it is crucial to know the shape and the sizes of the spoon. The same will be the case for the bouncing ball. The distorting problem is that during the inelastic collision, the boundary of the oval part of the ball will change continuousely. But the variation of the boundary depends on the physical model for the collision. So, you end up with a connected problem, the solution of the mathematical model which should give you the shape of the oval part of the ball depends on the boundary of the contact surface between the ball and the ground, and the boundary of the contact surface depends on the solution for the shape of the oval. This connected problems where the solution of two partial problems mutually depend on each other, is an example of a free boundary problem.

There are many other examples of free boundary problems occurring in nature, and even in our daily life. If you put an ice cube in a glass of water, heat will be transferred from the water to the ice, and little by little melting the ice. The diffusion of the heat is nicely described by the heat diffusion model where you have to consider the different heat capacity rate of ice and water. The boundary in this model is the surface of the ice cube, which you have to know to find an exact solution of the problem. But the boundary is not constant, in fact it is constantly changing, due to the ongoing melting process.

Again, we are faced with a connected problem; to solve the heat diffusion problem in water and ice, exact knowledge of the boundary of the ice cube is required. But to get this knowledge we must solve the heat diffusion equation under the instant boundary condition. The melting ice cube problem is known as the two-phase Stefan problem, named after Josef Stefan, a Slovenian physicist who introduced the general class of such problems around 1890.

In general, it is out of reach to give a complete solution of a free boundary problem. As a second-best, seen from a mathematical point of view, one would like to have some control about how the boundary develops during te process. For the Stefan problem this might also have a more general interest. If you scale up the ice cube in the glass of water, you can in fact face a global problem. An example is the Thwaites Glacier, nicknamed the Doomsday Glacier, in the Antarctic. Possible due to man-made climate changes the glacier is heading towards the sea. If the whole glacier should collapse into the ocean the sea level would globally increase by 65 cm. In addition, ice cubes of different sizes would for a period of time float around in the oceans. The size of the glacier is comparable to Great Britain and a possible collapse would definitely actualize a more thorough study of the Stefan problem.

In a paper from 1977; “The regularity of free boundaries in higher dimension”, Luis Caffarelli gave a ground-breaking contribution to the understanding of the boundaries of the free boundary problems. He showed that the shape of the boundary during the process will exhibit a certain mathematical regularity. For the melting ice cube we can think of this as that the surface of the ice will continue to be rather smooth, and not develop into some sort of porous medium.

THE REGULARITY OF FREE BOUNDARIES IN HIGHER DIMENSIONS

BY

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Introduction

The problem of studying the regularity of the free boundary that arises when considering the energy minimizing function over the set of those functions bigger than a given "obstacle" has been the subject of intensive research in the last decade. Let me mention H. Lewy and G. Stampacchia [14], D. Kinderlehrer [11], J. C. Nitsche [15] and N. M. Riviere and the author [5] among others. In two dimensions, by the use of analytic reflection techniques due mainly to H. Lewy [13], much was achieved.

Recently, the author was able to prove, in a three dimensional filtration problem [4], that the resulting free surface is of class C^1 and all the second derivatives of the variational solution are continuous up to the free boundary, on the non-coincidence set. This fact has not only the virtue of proving that the variational solution is a classical one, but also verifies the hypothesis necessary to apply a recent result due to D. Kinderlehrer and L. Nirenberg, [12] to conclude that the free boundary is as smooth as the obstacle. Nevertheless, in that paper ([4]), strong use was made of the geometry of the problem: this implied that the free boundary was Lipschitz. Also it was apparently essential that the Laplacian of the obstacle was constant.

In the first part of this paper we plan to treat the general non-linear free boundary problem as presented in H. Brezis–D. Kinderlehrer [2]. Our main purpose is to prove that if X_0 is a point of density for the coincidence set, in a neighborhood of X_0 the free boundary is a C^1 surface and all the second derivatives of the solution are continuous up to it. In the second part we will study the parabolic case (one phase Stefan problem) as presented by G. Duvaut [7] or A. Friedman and D. Kinderlehrer [9]. There we prove that if for a fixed time, t_0, the point X_0 is a density point for the coincidence set (the ice) then in a

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The front page of Caffarelli‘s 1977-paper

Caffarelli‘s 1977-paper became one of the starting points of an extensive research towards a better understanding of the regularity of solutions of a variety of mathematical models, including the Navier-Stokes equations, the Obstacle problem and the Monge-Ampere equation, to all of which Luis Caffarelli has given important contributions and been a leading star