THE ABEL PRIZE 2023

A Glimpse of the Laureate's Work

By Alex Bellos

The Abel Committee has awarded Luis A. Caffarelli the 2023 Abel Prize for his seminal contributions to the study of nonlinear partial differential equations. The text below gives a brief explanation of some of the work he has done in this area.

The discovery in the seventeenth century that the universe can be described by mathematical equations marked the beginning of modern science. Isaac Newton's second law of motion was an early example of such an equation: it states that the force on an object is equal to the mass of that object times its acceleration, usually expressed by the formula $F = ma$. Not only were Newton's laws a conceptual leap forward, but they also required a new type of mathematics.

This new mathematics, developed by Newton and Gottfried Leibniz, became popularly known as the infinitesimal calculus, and introduced the idea of an instantaneous "rate of change". Since these instantaneous rates of change are calculated by considering infinitesimal differences, equations in calculus became known as *differential equations.* They come in two classes: *ordinary differential equations*, which feature a single variable, and *partial differential equations (PDEs)*, which feature more than one variable.

PDEs are ubiquitous across science. They underlie our understanding of the physical world, beautifully modelling phenomena from heat to sound to electromagnetism to quantum mechanics. PDEs also arise in the social sciences, explaining the behaviour, for example, of epidemics, interest rates and stock options. Indeed, wherever there is a system involving multiple variables undergoing continuous change, you will find PDEs.

The power of a differential equation is that it predicts the future. For example, if I throw a ball I know – thanks to Newton's second law – that it will travel through the air in the shape of a parabola. I also know that if I throw the ball with a bit more power, or at a slightly different angle, that the ball will still travel in the shape of a parabola, even though it may be a slightly bigger or smaller one. In other words, Newton's second law is well-behaved: if I adjust the input values slightly, there are no nasty surprises in the output. The research of Luis Caffarelli asks similar questions of PDEs: when you adjust the input, do they also always act in the expected way, or are there values that trigger unstable, irregular or erratic behaviour?

Caffarelli's first area of investigation was the obstacle problem, a classic example in the field of nonlinear PDEs, which asks what is the equilibrium position when an elastic membrane pushes against a rigid obstacle, such as, for example, when a balloon presses against a wall. This work led him to the wider area of "free boundary problems", so-called because the boundary under discussion – such as where the membrane meets the obstacle, or where the balloon meets the wall – is unknown at the outset and is what needs to be determined. Other examples of free boundary problems include ice melting into water, or water seeping through a porous medium. In the case of ice melting into water, the free boundary is the interface between the ice and the water, and can be used to model other examples of phase transitions in physics, biology and finance.

Caffarelli revolutionised the study of free boundary problems in the 1970s, after which he turned his attention to probably the most famous PDEs in all mathematics, the Navier-Stokes equations. Formalised in the mid-nineteenth century, these two equations describe the motion of viscous fluids, such as how water flows down a stream or oil down a pipe. The first equation, marked i) below, states the fluid is incompressible.

$$\nabla \cdot \mathrm{V} = 0$$

i) Navier-Stokes equations – first equation

$$\rho \, \frac{DV}{Dt} = -\nabla p + \mu \nabla^2 V + \rho g$$

ii) Navier-Stokes equations – second equation

The second equation, marked ii), is an application of Newton's second law: it says that the mass times the acceleration (on the left of the equals sign) is equal to the force (on the right), which is broken down into the internal forces (pressure and viscosity) and external forces (usually gravity, hence the abbreviated *g*).

Physicists and engineers use the Navier-Stokes equations to predict the behaviour of fluid flows every day, and they work extremely well. Yet despite their practical importance, the equations are not fully understood. For example, it is an open question whether or not the equations are always 'smooth' or whether they will sometimes 'blow up', meaning that if you smoothly tweak the pressure, viscosity and so on, it is not known whether the velocities within the fluid will always change smoothly, or if the equations may throw up a point at which the velocity spikes to infinity. In the real world, velocity can never be infinity, so the discovery of singularities with infinite velocity would mean that the equations are somehow inadequate models of physical behaviour. The question of the smoothness of the Navier-Stokes equations has gained notoriety in recent years, since it is a Millennium Problem, one of the six problems the Clay Mathematics Institute has decided it will give a $1 million prize to the first person to provide a solution.

In 1982 Caffarelli, together with Robert Kohn and Louis Nirenberg, proved that if the Navier-Stokes equations do produce singularities, they will disappear instantly because the singularities produced cannot fill a curve in space time (meaning the three dimensions of space and the one dimension of time treated as four dimensions.) The 1982 paper remains the closest anyone has got to proving or disproving the smoothness of the Navier-Stokes equations, even after another four decades of intense research in this area.

PDEs arise when scientists try to describe natural laws, but they are studied by mathematicians for their own internal consistency and beauty. Luis Caffarelli has made his life's work the desire to establish that these tools have a rigorous mathematical foundation. He has been hugely influential in taming their wildness, making sure PDEs are meaningful representations of reality.