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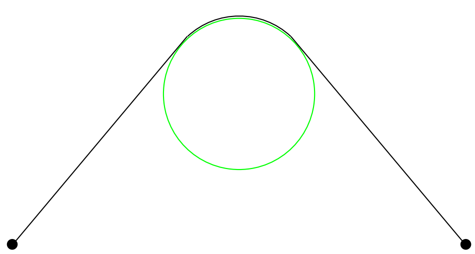
The Obstacle Problem

A stretched elastic rubber band will form a straight line. The reason is rather simple; the inherent property of a rubber band is to pull itself together, with the shortest path between the two endpoints as the result. The shortest distances between two points is along a straight line.



Now put the rubber band on top of a ball and stretch it down on each side. The shape of the elastic band will have three parts. On straight line from your hand to the ball, a curved part along and over the ball and another straight line from the ball to your other hand. Using a little bit of mathematics we can easily compute the shape of the band; it will leave the ball exactly at the point where the tangent line of the ball is targeted directly towards your fingers. If we know the position of the hands and the position and size of the ball, the computation is straightforward. Again you use the fact that the shortest distance between two points is the straight line.

If you prefer to shoot at sparrows with a cannon you can look at the following argument for the fact "The shortest distance between two points is the straight line".

Let P and Q be two points in the plane, and let

$$C : \mathbf{r}(t) = (x(t), y(t))$$

be a parametrized curve connecting the two points. Basic calculus says that the arc length of the curve is given by the integral

$$\int_C \sqrt{dx^2 + dy^2} = \int_C \sqrt{1 + (y')^2}dx$$

The shortest curve is therefore the solution to the problem of finding the least value of the above integral as C varies over all curves connecting the points P and Q. Inserting the Lagrangian $L(x, y, y') = \sqrt{1 + (y')^2}$ of this optimization problem into the Euler-Lagrange equation, we get

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = -\frac{d}{dx}\frac{y'}{\sqrt{1 + (y')^2}} = 0$$

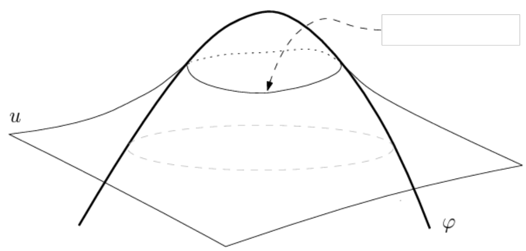
i.e. $\frac{y'}{\sqrt{1+(y')^2}} = C$ for some constant C. Solving for y' we get

$$y'(x) = \frac{C}{\sqrt{1 - C^2}}$$

A function with a constant derivative is linear, i.e. the graph is a straight line.

Extending this problem to two dimensions we replace the rubber band with a non-breakable soap film. The boundary of the soap film is any space curve, and the obstacle is again a ball. The soap film has the same property as the rubber band, to pull itself together, constituting a surface of minimal area. If the boundary is suitably nice and the obstacle is removed, this problem can be solved. Solving this problem means that we can find a formula for the resulting surface.

Reintroducing the obstacle, or more literally, pushing the ball against the non-breakable soap film, we challenge a similar problem as with the rubber band. The outer edge of the soap film is attached to the frame, in the inner part there is a region where the film is in complete contact with the ball, this is the part we call the contact surface. And finally, between the contact surface and the frame, the soap film will try to complete its deepest wish; to minimizie its surface.



As we saw in the non-obstacle case, when the boundary is sufficiently nice, it is possible to find a solution to the minimal surface problem. When introducing the obstacle, we still have a minimal surface problem, but now the boundary has two components, in addition to the outer edge there is an inner edge; the boundary of the contact surface, referred to as the free boundary. A major challenge in solving this problem is that this free boundary is not fixed. If it was fixed we could solve the minimal surface problem. But the free boundary is in a way a consequence of the forming of the minimal surface. Thus we are faced with a connected problem; The solution for the minimal surface problem is highly dependent of the free boundary, but the shape of the free boundary depends on the formation of the minimal surface.

Luis Caffarelli has made important contributions to the understanding of the obstacle problem, in particular in the study of the free boundary. In the two cases we have looked at, the one-dimensional rubber band and the two-dimensional soap film, our intuition might tell us that the free boundary is rather smooth. If there there had been any sharp edge or fold, a natural first thought is that the elasticity of the rubber band or the the contractibility of the soap film would stretch out and smoothen everything. It is often said that a soap film is wiser than a mathematician, meaning that the soap film immediately computes the shape of the minimal surface, while the mathematician at best can find the solution after hours of heavy work. In this case Caffarelli has done this heavy work once and for all, proving that the free boundary in fact will be rather smooth.The really difficulty in proving such a result is the lack of explicit description of the free boundary. How can you prove anything about an object you don‘t know anything about but its existence? Well, that is one of the reasons for the choice of Luis Caffarelli as the 2023 Abel Laurate, he was able to complete this seemingly impossible task.

The obstacle problem is not limited to rubber bands or soap films. Other examples of applications include the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, and financial mathematics.