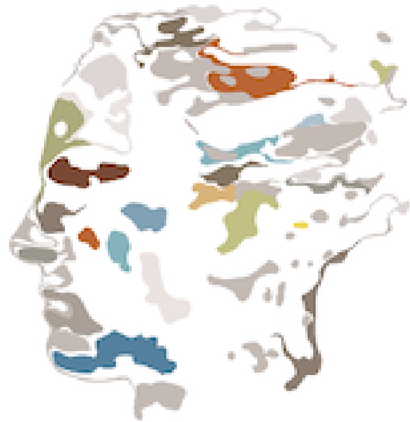


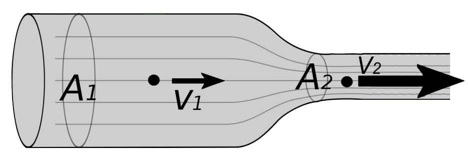
Luis Caffarelli
Abel Prize Laureate 2023

Mathematics of Mathematical Models

If you predict that today's weather will be the same as yesterday's weather, you will succeed with a probability of approximately 0.5. If you in addition incorporate some old weather sayings, like "Red sky at night, sailors delight. Red sky in morning, sailors take warning," you will noticeably increase your fortune-telling abilities. But still, if you plan to cross the sea, you would probably prefer a more knowledge-based forecast. So, you look up at the Internet or listen to your radio to read or hear what the meteorologist can tell you about challenges you will face on your imminent journey.

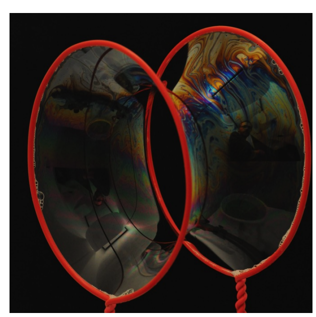
Putting on your sunglasses and rejoice that the wind is blowing just right in the mainsail, you send a warm thought to the meteorologist who fed her computer with a lot of data and some physical laws to give you a close to perfect description of today's nice weather.

Weather forecast, as well as many other models for how nature behaves, is based on what is called a "partial differential equation", or a PDE for short. The general setup for a PDE as a mathematical model for something we observe in nature, is the correspondence between an acting force and a resulting reaction. The reaction can either be described as a time-based process, or just as a geometric configuration with no time involved. The acting force can originate from different sources, a gravitational field, pressure or a temperature gradient. In additional to the source of the acting force, the geometry of the object in study plays a major role.



Source: Rune Mathisen. (https://ndla.no/article/10454)

The behaviour of a gas flowing through a pipe is highly dependent of the shape of the pipe. If we narrow the pipe, the gas will flow faster, and obstacles will cause turbulence. Or, dipping a metal frame into a soapy water, suitable for blowing soap bubbles, the soap film will form a surface of minimal area with the shape of the frame as the only constraint.



Source: Soapbubble.dk

The gas flow through a pipe is modelled by what is called the Navier-Stokes equations. The equations come from applying Isaac Newton's second law to fluid motion, together with assumptions about how the molecules in the fluid interacts.

A solution of the Navier-Stokes equations is a velocity field, describing the velocity of the fluid at any time and any point. Boundedness of this solution means that the speed of any part of the fluid will never exceed a certain given value. In the physical world this not an issue since infinite speed is an impossibility. But we can still ask the truly physical question; does there exist a limit for the speed of a tornado, or will we constantly observe new speed records? Mathematically this question is rephrased as a question of boundedness of solutions. Do the Navier-Stokes equations develop unbounded solutions in finite time? This question is called the Navier-Stokes existence and smoothness problem. The Clay Mathematics Institute has called this problem one of the seven most important open problems in mathematics and has offered a US$1 million prize for a solution or a counterexample.

The Navier-Stokes equations involve time as a parameter, i.e. the equation models what happens in time based on acting forces and geometry. An example of a physical phenomenon which is modelled by a PDE, but which not involves time as a parameter, is the minimal surface phenomenon. When you dip a steel wire frame in soapy water, if you are lucky, you will catch a soap film attached to the frame. Since the soap molecules prefer to stick together as much as possible, the soap film will always exhibit a minimal surface. This means that provided the film is attached to the frame, it is not possible to find a surface of smaller area.

Even if we consider the formation of the minimal surface as a purely geometric result, there is of course a time-dependent process that takes place after dipping the steel wire frame in the soapy water. The soap molecules work fast, reaching their final position in just a fraction of a second. In the process they move a bit around, playing their game with all the other molecules, and finally settle down. They have reached a state of equilibrium. Thus, the solution of the time-independent model can also be considered as the equilibrium state of a time-dependent process.

A common problem for all mathematical models is to find solutions. Suppose you put up a mathematical model for something that takes place in nature and you are clever enough to find solutions to the involved equations. Then you actually will be in position to predict something about the future based on scientific reasoning. Your predictions will be far more valuable than if you were just guessing.

In general it is very difficult to find the solutions. In many cases it is hard even to prove that there exist solutions. And if you know that there exists solutions, they may have bad properties. Luis Caffarelli has dedicated his professional life to study the nature of the solutions of various partial differential equations. Thanks to his efforts and to the contribution of many other mathematicians, our insight in the nature of the solutions has increased significantly over the last 50 years. Tthe Abel Prize committee writes in their citation for this year's Abel Prize: "Combining brilliant geometric insight with ingenious analytical tools and methods, he [Caffarelll] has had and continues to have an enormous impact on the field".