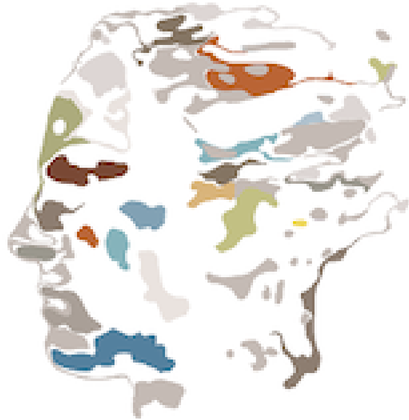

Michel Talagrand
Abel Prize Laureate 2024

Concentration of measure

A disk of radius 1 (unit disk) can be described as the set of points (x, y) in the plane such that $x^2 + y^2 \leq 1$. The area of the unit disk is very well known to be π. Inside the unit disk we place a concentric disk of radius $1 - \epsilon$ for some arbitrary small positive number ϵ. The area of this smaller disk is $\pi(1-\epsilon)^2$ and the area of the annulus is $\pi(2\epsilon - \epsilon^2) \approx 2\pi\epsilon$. As a fraction of the total area the inner disk counts for $\frac{\pi(1-\epsilon)^2}{\pi} = (1-\epsilon)^2$. If we do the same construction for a sphere, given by $x_1^2 + x_2^2 + x_3^2 = 1$, the fraction of the inner sphere of radius $1 - \epsilon$ vs. the unit sphere is $\frac{\frac{4}{3}\pi(1-\epsilon)^3}{\frac{4}{3}\pi} = (1-\epsilon)^3$ and for a 3-sphere of radius R and volume $\frac{1}{2}\pi^2 R^4$, the fraction is $\frac{\frac{1}{2}\pi^2((1-\epsilon)R)^4}{\frac{1}{2}\pi^2 R^4} = (1-\epsilon)^4$. The corresponding fraction for a d-sphere, given as the zero set of the equation $x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1$ in the Euclidian space $\mathbb{R}^{d+1}$ is $(1-\epsilon)^{d+1}$. The consequence is that for a high-dimensional sphere, where we let d grow towards ∞, almost all of its volume will be concentrated in a thin annulus.

In a similar way one can show that for high-dimensional spheres almost all of its surface will be concentrated in a small belt around equator.

An other strange phenomenon in high-dimensional geometry conceretns the volume of a d-sphere of radius 1. The volume is proportional to $\pi^{\lfloor \frac{d}{2} \rfloor}$, which grows without any limit when d increases.

A high-dimensional onion has almost all its content in the outer shell.
Source: frukt.no

In the citation the Abel comitee says: ”One of the threads running through Talagrand´s work is to understand geometric properties of a high-dimensional phenomenon and to crystallise this into sharp estimates with broad scopes of applicability.” The example above concerning high-dimensional spheres goes into the core of this quote.

We can give a more general description of the ”concentration of measure phenomenon”: Suppose we have given a large number of ”nice” random variables, and consider their sum. If the random variables are sufficiently independent, the sum will sharply concentrate in an interval which is much narrower than one should expect by the first glance.