THE
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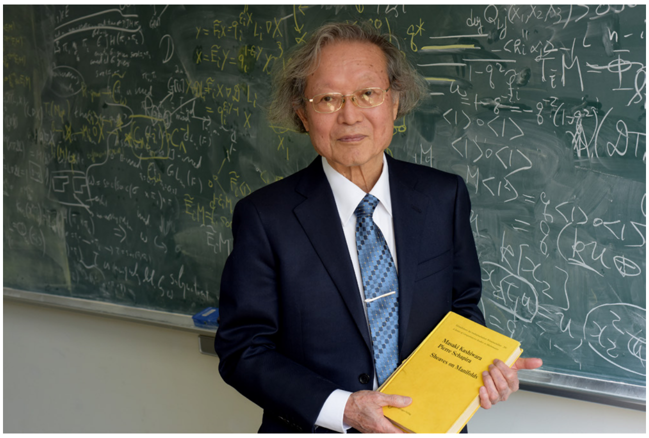
Masaki Kashiwara's work – for non-mathematicians

By Timandra Harkness



Masaki Kashiwara: Peter Bagde / Typos1 / The Abel Prize

Masaki Kashiwara is a builder of bridges between different fields of mathematics. Not just little bridges that join two ideas, but spectacular bridges that span mathematical worlds. Imagine a beautiful bridge that uses completely new construction techniques to join Norway with South America, or Japan with Antarctica.

Kashiwara connected the mathematical continents of algebra and analysis, and then the third mathematical continent of geometry, with his original thinking. His ideas are not just beautiful and brilliant in themselves, they open up routes for many other mathematicians to explore new territories and solve new problems.

Masaki Kashiwara's love of algebra began with turtles and cranes. At school, he was set a common problem – Tsurukamezan. If the number of heads is X, and the number of legs is Y, how many cranes

Tsurukamezan: Turtles and cranes: Crediting: DALL-E

and turtles are there? If there are two heads and six legs, for example, there must be one crane and one turtle.

More than solving the problem, Kashiwara enjoyed finding a generalised method that would solve it every time, for any X and Y. He has brought this algebraic approach to other fields of mathematics throughout his career.

At Tokyo University, with his supervisor and mentor, Mikio Sato, Kashiwara pioneered algebraic analysis, applying the methods of algebra to the problems of analysis – the mathematics of how things change.

Masaki Kashiwara and Mikio Sato:
Private photo by Tetsuji Miwa / RIMS / Kyoto University

Kashiwara's master's thesis, written when he was only 23, established the foundations of D-Module theory, as a way to algebraically analyse systems of linear partial differential equations.

Differential equations describe how things change. You may remember solving differential equations at school to answer questions like: how fast is this car moving at a particular point? Is it speeding up or slowing down? Mathematicians like Kashiwara work with systems of linear partial differential equations. They are less interested in solving equations, and more interested in finding out about the properties a solution will have – if there even IS a solution.

Not every point in every differential equation has a definable solution. If a function is defined by 1/x, then when x is zero, 1/x would be… infinity. A point like this is called a singularity. One longstanding problem in mathematics, Hilbert's 21st problem, asks about the solutions of particular kinds of differential equations that have such singularities in the complex domain, where real numbers and imaginary numbers live together.

On a complex manifold, the solutions around these singularities can behave very strangely indeed. Following the solutions around a singularity can take you back to where you started, only to find that the solutions are behaving differently this time around. A system of differential equations that have these strange points is called monodromic.

Hilbert's 21st problem (also known as the Riemann-Hilbert Correspondence) asked: can we say that a particular type of system of differential equations will *always* have this property of monodromy, and can we predict exactly *where* the strangely-behaving points of singularity will appear?

David Hilbert (1912):
Unknown photographer / Postcard from
University of Göttingen / Possible source:
Reid, Constance (1970) Springer Publishing /
Public domain

Using his D-Module Theory, Kashiwara was able to show that, in any dimension, there will always be a unique differential equation that fulfils the predicted requirements[1].

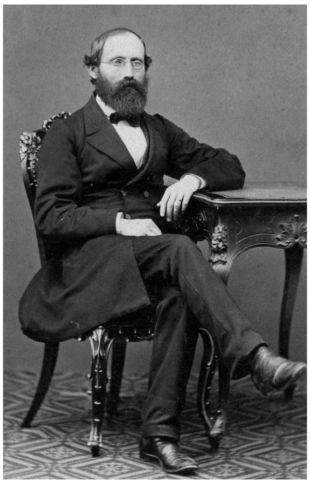
In proving this Riemann-Hilbert Correspondence, Kashiwara built a new bridge to the field of topology, relating D-Modules to topological objects called sheaves. Kashiwara's work on sheaves, collaborating for over 50 years with Pierre Schapira, also built a bridge to yet another mathematical region – representation theory.

Representation theory uses algebra to study symmetry.

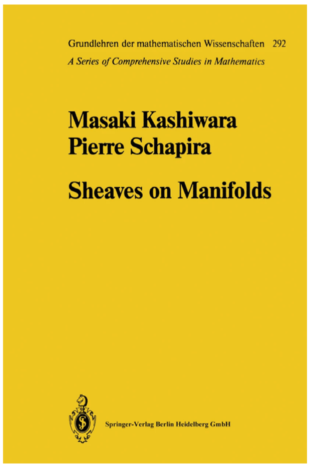
We are familiar with different kinds of symmetry in everyday objects. A square tile on a flat surface has a number of symmetries. You can rotate it through a quarter, half or full turn; you can reflect it in a line between opposite corners, or a line joining the halfway

[1] Zoghman Mebkhout independently showed this at the same time.

points of opposite sides. You can slide across a tiled surface and find the pattern repeated – translational symmetry.



Sitting: Bernhard Riemann: From the Family Archive of Thomas Schilling / Public Domain.



Sheaves of Manifolds: Springer Publishing

Pierre Schapira: Photo: Christian Rodrigues / ICM 2018 / Public domain

A three dimensional object like a cube has all these symmetries and more, because it can rotate, reflect or slide in three dimensions. Mathematical objects in spaces with more dimensions can have even more orders of symmetry.

Relationships between the different symmetries of an object can be expressed by a branch of algebra called group theory. Niels Henrik Abel, the Norwegian mathematician for whom the Abel Prize is named, did important work in this field and gave his name to Abelian groups.

Niels Henrik Abel
Drawing by av Johan Gørbitz.
Owner: Department of Mathematics,
University of Oslo

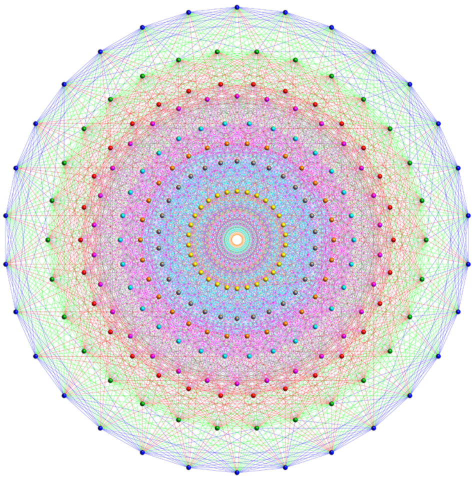
Squares, cubes and many other familiar objects have a finite number of symmetries. But other objects, like a circle or a sphere, have an infinite number of symmetries. You can rotate a sphere on any axis, or reflect it in any plane through the centre. Norwegian mathematician Sophus Lie[2] invented Lie groups, which describe such continuous groups of symmetries.

Quantum groups were first developed in physics, in order to use Lie groups in statistical mechanics, but Kashiwara invented another completely new way to use them: crystal bases.

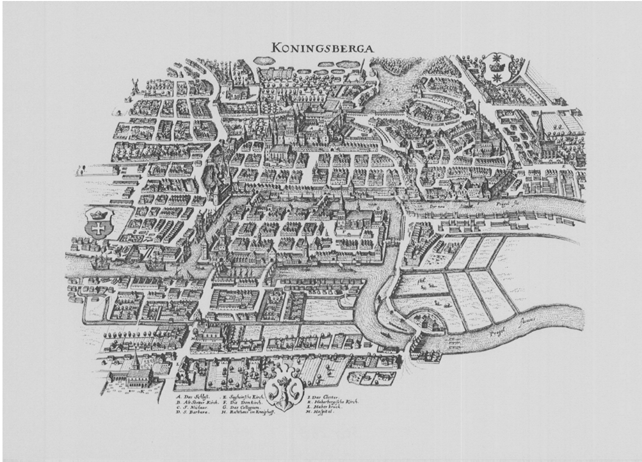
Kashiwara's invention of crystal bases built another bridge, this time between representation theory and graph theory.

[2] Sophus Lie first tried to set up the Abel Prize in 1902, on the 100th anniversary of Abel's birth.

Sophus Lie: Photo by Ludwik Szacinsky / Oslo Museum / Public domain (CC)



Lie group: Emulation of hand drawn original by Peter McMullen / Wikimedia Commons



Kõningberg Bridge: Alamy Stock Photo

Leonard Euler: Painting (1778) by Joseph Frédéric Auguste Darbès / MAH Museum of Art and History, Geneve / CC 3.0

Graph theory uses nodes, or vertices, connected by edges or links, to model structures and systems. The Swiss mathematician Leonhard Euler invented graph theory in 1736, using it to solve a puzzle called the Königsberg Bridge problem.

The city of Königsberg on the Baltic sea – now called Kaliningrad – has two islands in the middle of its river, and seven bridges joining the islands with the two banks, and with each other. The puzzle was to find a route that would cross each bridge exactly once, bringing the walker back to the starting point.

Euler used a graph to represent the bridges as seven links and the places they connected – two banks and two islands – as four nodes. By counting the number of links attached to each node, he was able to show that it would never be possible to find such a route. This abstraction of problems to simplified structures of nodes and links is now used in computer science, chemistry, and even in printing coloured maps.

Kashiwara's crystal basis turns the combined symmetries of a mathematical object, as described by a quantum group, into a graph. All objects with the same combinations of symmetries share the same graph. If the object has two different symmetries, like a corkscrew which combines rotation and translation, the graph will have two unconnected sections.

Crystal bases are now used throughout representation theory.


Masaki Kashiwara: Peter Bagde / Typos1 / The Abel Prize

The 2025 Abel Prize is awarded to Masaki Kashiwara for his work on D-modules and crystal bases in particular, but also for his wider contributions to Algebraic Analysis and Representation Theory, and to mathematics in general.

Pioneer, visionary, and builder of beautiful mathematical bridges, Masaki Kashiwara.