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The Riemann-Hilbert Correspondence

David Hilbert (1862-1943)

Source: Wikipedia

At the International Congress of Mathematicians in Paris in 1900 the German mathematician David Hilbert put forth a list of 10 unsolved problems in mathematics. Later he published an extended list including 23 problems, all considered to be very influential for 20th-century mathematics. Some of the problems are still open, some has been solved.

Hilbert´s 21st problem

The 21st problem, already included in the first list of 10 problems, concerns the proof of the existence of linear differential equations having a prescribed monodromy group. In his description of the problem, Hilbert suggests that Bernhard Riemann was well aware of this problem. Thus, the problem has later been referred to as the Riemann-Hilbert problem. The problem has led to several bijective correspondences, known as Riemann-Hilbert correspondences. A Riemann-Hilbert correspondence establishes an equivalence between different categories, telling us that to find a solution of the Riemann-Hilbert problem in one setting is equivalent to finding a solution in a somewhat different setting. Variations of the Riemann-Hilbert problem have been proved, but the original form, as Hilbert sated it, has been proved to be wrong.

Hilbert presented his 21st problem as follows:

”In the theory of linear differential equations with one independent variable z, I wish to indicate an important problem one which very likely Riemann himself may have had in mind. This problem is as follows: To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group. The problem requires the production of n functions of the variable z, regular throughout the complex z-plane except at the given singular points; at these points the functions may become infinite of only finite order, and when z describes circuits about these points the functions shall undergo the prescribed linear substitutions.”

Differential equations

A differential equation is an equation that gives a relation between a function and its derivatives. Differential equations have been studied since the seventeenth century and they constitute an important mathematical tool for understanding phenomena in nature. Examples of famous differential equations are the heat equation and the wave equation. The first one describes the propagation of heat through a material when exposed to temperature differences, the other one describes how waves are rolling over the sea.

Using basic laws of nature and knowledge of the behaviour of liquids, we can suggest a differential equation that describes what happens in the bathtub when we remove the stopper. The solution of the equation describes a whirlpool motion, close to what is observed in the liquid. The difference between the mathematical model, expressed by the differential equation, and the real motion of the liquid, increases as we approaches the centre of the whirlpool. In the bathtub there is no water at all in the centre of the whirlpool, it has already been drained out. In the model, however, the speed of the circular motion of the water will increase as we approach the centre. In the centre the model will collapse. The model has a singularity at this point.

Monodromy - changing the solution upside-down

Singularities have an interesting effect on the solutions of the differential equation. A mathematical phenomenon called monodromy may occur. The word monodromy is of Greek origin and means ”running round simply” Now, suppose we have found a solution to the differential equation. The value of the function will vary continuously along arbitrary curves. When we return to the starting point, the value of the function will be the same as when we started. i.e., as long as we do not run along a path encircling the singularity. In that case the value might change. This is what is called monodromy. It is a bit like motions in a spiral staircase. As long as the full loop does not encircle the centre of the staircase, we remain at the same level, but encircling the centre will bring us either upwards or downwards.

As an example, consider the differential equation

$$z\frac{df}{dz} = \frac{1}{2}f$$

defined on the punctured complex plane, $\mathbb{C} \setminus \{0\}$. If we write $z = re^{i\theta}$ we see that

$$f(z) = f(re^{i\theta}) = \sqrt{re^{i\theta}} = \sqrt{r}e^{i\frac{\theta}{2}}$$

is a solution to the equation on a disc of radius 0.99 around $z = 1$. Notice that the choice of radius 0.99 is just to avoid the singularity located at the origin. Now consider four functions, $f_k(z) = f(z)$, $k = 0, 1, 2, 3$ defined on discs of radius 0.99 around 1, i, -1 and $-i$, respectively. Consequetive discs intersect, and the functions agree on the intersections. But nevertheless we have

$$f_3(1) = f_3(e^{2\pi i}) = e^{i\pi} = -1 \neq f_0(1)$$

which shows that the solutions of the given differential equation have non-trivial monodromy.

Riemann-Hiilbert correspondence

A differential equation can have different numbers and types of singularities, and also different types of monodromy. Hilbert was well aware of this fact. What he wondered about was the opposite problem: Given the monodromy and the singularities, can we always find a differential equation? The problem has elicited many answers during the 20th century. At the same time the problem has been generalised in many directions. In the original setting the problem takes place on a Riemann sphere. In a slightly more setting the Riemann sphere is replaced by a general Riemann surface, and moving to higher dimensions we need to consider even more general complex manifolds. With such level of generality for the underlying space the differential equation has to be replaced by the more general term of a connection on the manifold. A proof of the Riemann-Hilbert correspondence for algebraic connections with regular singularities is due to Pierre Deligne. The award of the Abel Prize to Deligne in 2013 was partly based on this result. Masaki Kashiwara provided in the early 80s a proof in the even more general setting of regular holonomic $\mathcal{D}$-modules. An alternative proof of Kashiwara´s theorem was independently, and almost at the same time given by Zoghman Mebkhout.

To illustrate a version of the Riemann-Hilbert correspondence we have to introduce a bit more complicated mathematical machinery. Let M be a differentiable manifold. There is a functor from the category of vector bundles on M with flat connections to the category of local systems on M, given by

$$(V, \nabla) \mapsto V^{\nabla} = \ker(\nabla)$$

where V is the vector bundle and ∇ is the flat connection. The functor gives an equivalence of categories.

As an example of this version of the Riemann-Hilbert correspondence, consider the trivial bundle $\mathcal{O}_X^2$ on $X = \mathbb{G}_m$ with connection

$$\nabla \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = d\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \frac{dz}{z}$$

The kernel of this connection is the solution to the system

$$df_1 = 0$$
$$df_2 - f_1\frac{dz}{z} = 0$$

It can be shown that the solution is given by

$$f_1 = B$$
$$f_2 = B\log z + A$$

or

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ \log z \end{pmatrix}$$

This is an example of a local system of rank 2 and it corresponds by the Riemann-Hilbert correspondence to the pair (V, ∇).

Notice that a local system $\mathcal{L}$ on a topological space X is the same as a locally constant sheaf, i.e. a sheaf where all stalks are the same.

Let $P : \mathcal{O}_X \to \mathcal{O}_X$ be the map $P = z\frac{\partial}{\partial z} - \frac{1}{2}$, where $X = \mathbb{C}$ and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. The map P is closely related to the differential equation

$$z\frac{df}{dz} = \frac{1}{2}f$$

defined above. The kernel of P is given by $f(z) = C\sqrt{z}$ for any complex number C. For any point $z \in \mathcal{C} \setminus \{0\}$ the set of solutions to the equation is 1-dimensional. Thus the kernel $\mathcal{L} = \text{ker}(P)$ is locally constant ($= \mathbb{C}$) on $X \setminus \{0\}$, but not on all of X ($f(0) = 0$).

Denote by L the local constant stalk of the local constant sheaf $\mathcal{L}$. The monodromy of the solutions to the equation $P = 0$ now gives a representation

$$\pi_1(X, x) \longrightarrow \text{Aut}(L)$$

of the fundmental group of X on the group of automorphisms of the stalk L, called the monodromy representation of $\mathcal{L}$ and adding yet another equivalence between categories to the list of Riemann-Hilbert correspondences.