THE ABEL PRIZE

Masaki Kashiwara
Abel Prize Laureate 2025

“For fundamental contributions to algebraic analysis”

Source: Kyoto University

Masaki Kashiwara, Research Institute for Mathematical Sciences, Kyoto University, Japan is awarded the Abel Prize for 2025,

“for his fundamental contributions to algebraic analysis and representation theory, in particular the development of the theory of $\mathcal{D}$-modules and the discovery of crystal bases.”

$\mathcal{D}$ for differential

The Abel Committee states in the citation: “$\mathcal{D}$-modules provide an algebraic language for studying systems of linear partial differential equations. The 1970 master’s thesis of Kashiwara develops the theory of analytic $\mathcal{D}$-modules, introducing the fundamental notion of characteristic variety, and proving a vast generalization of the Cauchy-Kovalevskaya theorem. This demonstrated early on the power of algebraic methods in tackling problems of an analytic nature.”

The Airy equation is named after Sir George Biddell Airy (1801–1892), a royal astronomer at Cambridge. The Airy equation is an ordinary differential equation;

$$\frac{d^2y}{dx^2} - xy = 0$$

with several applications. The equation has two independent non-elementary solutions called the Airy and the Bairy functions (notice the mathematical humorous naming). We can transform the Airy equation into a system of linear partial differential equations, using the algebraic language of $\mathcal{D}$-modules. A necessary step is to introduce the Weyl algebra;

$$W = \mathbb{R}\langle x, y\rangle / (xy - yx + 1)$$

in two non-commuting variables x and y, named after the German mathematician Hermann Weyl (1885–1955), and introduced to study the Heisenberg uncertainty principle in quantum mechanics. A $\mathcal{D}$-module is a module M over the Weyl algebra, i.e. a vector space over $\mathbb{R}$ with an action of the Weyl algebra.

As an example we consider the $\mathcal{D}$-module structure of the ordinary polynomial ring in one variable; $M = \mathbb{R}[x]$. The two variables x and y of W acts very differently on M; as x acts by ordinary multiplication, the element y is more like a derivation operator. In fact, it would have been more appropriate to write $y = \frac{d}{dx}$, and as a consequence we have

$$y \cdot f(x) = f'(x)$$

The relation $xy - yx + 1 = 0$ reflects the product rule for derivation. To see this, write the relation as $yx = xy + 1$

and consider its action on a polynomial $f(x)$. By substituting $y = \frac{d}{dx}$ in the equation we get

$$\frac{d}{dx}xf(x) = x\frac{d}{dx}f(x) + f(x)$$

which corresponds exactly to the product rule for derivation.

Next we transform the Airy equation into a W-linear operator

$$\phi : W^2 \to W^2$$

by letting

$$\begin{pmatrix} f \\ g \end{pmatrix} \mapsto \begin{pmatrix} y & -1 \\ -x & y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$

Let V be the cokernel of the map ϕ.

A solution of the Airy equation in the polynomial ring $\mathbb{R}[x]$ is equivalent to an element in $\mathrm{Hom}_W(V, \mathbb{R}[x])$, given by a vector $(f, g) \in \mathbb{R}[x]^2$ such that

$$\begin{pmatrix} y & -1 \\ -x & y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. $g = yf = f'$ and

$$yg - xf = g' - xf = f'' - xf = 0$$

which is precisely the original Airy equation. This gives a nice example of how a problem of analysis is transformed into an algebraic setting, by the $\mathcal{D}$-module technology.

Kashiwara also used the theory of $\mathcal{D}$-modules to solve other problems. The Abel committee mentions some of these achievements in the citation:

"Kashiwara formulated and proved a vastly generalized Riemann-Hilbert correspondence, i.e., the equivalence between regular holonomic $\mathcal{D}$-modules and perverse sheaves (obtained independently by Zoghman Mebkhout)."

"The Kazhdan-Lusztig conjecture in representation theory can be viewed as connecting characters of representations to intersection cohomology groups. It was proved by Kashiwara together with Jean-Luc Brylinski, in a striking application of the Rieman-Hilbert correspondence."

The beauty of a crystal base

In addition to $\mathcal{D}$-modules, the Abel committee highlights the notion of crystal base: "Inspired by the study of solvable lattice models in mathematical physics, Vladimir Drinfeld and Michio Jimbo independently formalized quantum groups in the late '80s. Quantum groups are deformations of the universal enveloping algebras of complex semi-simple or Kac-Moody Lie algebras. Kashiwara introduced the notion of crystal bases and proved the existence of crystal bases for integrable highest weight representations of quantum groups."

Lie algebras are named after the Norwegian mathematician Sophus Lie (1842-1899). Inspired by Niels Henrik Abel´s work on symmetries of algebraic equations, Lie studied continuous symmetries of differential equations, referred to as transformation groups. Later on the groups have been called Lie groups. An important example of a Lie group is SL_2, the set of 2×2-matrices of determinant 1 over some appropriate ground field. The set has a group structure, meaning that we can multiply matrices of determinant 1 and the product will still have determinant 1. At the same time SL_2 is the space of all 4-vectors $(a_{11}, a_{12}, a_{21}, a_{22})$ where $a_{11}a_{22} - a_{12}a_{21} = 1$, which defines a 3-dimensional manifold. So a Lie group is at the same time a group and a manifold, and the two structures are closely tied together.

To any Lie group there is associated a Lie algebra, describing the tangent space structure of the Lie group. The corresponding Lie algebra of the Lie group SL_2 is denoted $\mathfrak{sl}_2$, and it is given by

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, a_{11} + a_{22} = 0 \right\}$$

An important part of the structure of a Lie algebra is the bracket $[-,-] : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{sl}_2$. The abstract definition of the bracket is modelled on the more familiar commutator notion $[a, b] = ab - ba$. It satisfies the equalities $[a, a] = 0$, $[a, b] = -[b, a]$ and a third equality called the Jacobi identity. The Lie algebra $\mathfrak{sl}_2$ is a 3-dimensional vector space, but we only need two generators to describe it as a Lie algebra. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = [e, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The three elements $\{e, f, h\}$ constitutes a basis for the underlying vector space, but for the Lie algebra, with the bracket included in the definition, only the two generators e and f are needed.

To any Lie algebra $\mathfrak{g}$ there is an associated object, called the universal enveloping algebra of the Lie algebra, denoted $U(\mathfrak{g})$. In the universal enveloping algebra we introduce multiplication as part of the structure, consistent with the bracket operation, i.e. $[e, f] = ef - fe$. The universal enveloping algebra carries the same information as the Lie algebra, but the additional multiplication structure opens for more flexibility in handling the object.

The universal enveloping algebra $U(\mathfrak{g})$ contains some invisible secrets. To enlighten the secrets we let the universal enveloping algebra be part of a more general structure by introducing a new type of object, called a quantum group, denoted $U_q(\mathfrak{g})$. A quantum group has a built-in parameter q, and when specialzing the parameter to $q = 1$, the quantum group reduces to the universal enveloping algebra $U(\mathfrak{g})$. The secrets we are looking for can be found in $U_q(\mathfrak{g})$ and be understood in $U(\mathfrak{g})$ by studying what happens when $q \to 1$. The symbol q is the main ingredient in this quantization process, playing the same role in a mathematical context as Planck´s constant $\hbar$ plays in quantum mechanics.

The quantum group $U_q(\mathfrak{sl}_2)$ is no more a Lie algebra, but has a richer structure, mathematicians would call it a Hopf algebra. The quantum group is generated by t, t^{-1} in addition to e and f, subject to the relations:

$$tet^{-1} = q^2 e$$
$$tft^{-1} = q^{-2} f$$
$$[e, f] = \frac{t - t^{-1}}{q - q^{-1}}$$

If we let $t = q^h$ and use the approximation $e^x \approx 1 + x$, the limit as $h \to 0$ (i.e. $q \to 1$) returns the relations of the original Lie algebra.

$$[h, e] = 2e, \qquad [h, f] = -2f \qquad [e, f] = h$$

A general tool to study intricate structures like the quantum group $U_q(\mathfrak{sl}_2)$ is to consider its representations. The idea of a representation is to substitute elements in the quantum group by matrices. The representation does not have to be faithfull, meaning that more elements can be replaced by the same matrix, but nevertheless the representations will illustrate the structure or at least parts of the structure of the object it represents.

It has been a long-lasting challenge to find suitable bases for representations of Lie algebras. Suitable in the meaning, easy to handle and mirroring the interesting and maybe even the secret properties of the Lie algebra. Kashiwara presented in an article from 1990 a way of constructing a basis for a representation, called a crystal base. The construction goes via an appropriate basis for a representation of the associated quantum group. When we let $q \to 1$ the basis is, by construction, also a basis for the representation at $q = 1$, i.e. a representation of the Lie algebra $\mathfrak{sl}_2$. Kashiwara´s result ensures that such nice bases exist and that they have the right properties. As a bonus the constructed bases can be given a rather simpel combinatorial description, using so-called Young diagrams, a concept introduced by Alfred Young at Cambridge University, already in 1900.