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## Crystal bases

The Abel Committee writes in the citation: "Inspired by the study of solvable lattice models in mathematical physics, Vladimir Drinfeld and Michio Jimbo independently formalized quantum groups in the late '80s. They are deformations of the enveloping algebras of complex semi-simple or Kac-Moody Lie algebras. Kashiwara introduced the notion of crystal bases and proved the existence of crystal bases for integrable highest weight representations of quantum groups. The proof, which proceeds by an intricate induction process now known as the grand loop argument, is a tour de force that has not been much simplified over time.

Kashiwara also generalized crystal bases to global bases, which were independently discovered by George Lusztig under the name canonical bases. This work can be thought of as a vast and fruitful generalization of the theory of Young diagrams and Young tableaux."

### The quantum group $U_q(\mathfrak{sl}_2)$

Consider the vector space of $2 \times 2$-matrices of trace 0 over some appropriate field $K$, i.e. the set

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in K^{2\times 2} \mid a_{11} + a_{22} = 0 \right\}$$

A basis for the vector space is given by the three matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Trace of a $2 \times 2$-matrix $a$ is the sum of the diagonal elements, tr$(a) = a_{11} + a_{22}$.The product $ab$ of two matrices $a$ and $b$ of trace 0 does not necessarily have trace 0, but the bracket $[a, b] = ab - ba$ has, since tr$(ab) =$ tr$(ba)$.

The vector space generated by $e$, $f$ and $h$ of trace-less matrices is therefore closed under the bracket operation. A vector spaces with this property is called a Lie algebra, named after the Norwegian mathematician Sophus Lie (1842-1899). The Lie algebra with the underlying vector space generated by $\{e, f, h\}$ is denoted $\mathfrak{sl}_2$; it is the tangent space of the special linear group, i.e. $2 \times 2$-matrices of determinant 1.

To any Lie algebra $\mathfrak{g}$ there is an associated object, called the universal enveloping algebra of the Lie algebra, denoted $U(\mathfrak{g})$. In addition to the structure of the Lie algebra the universal enveloping algebra is equipped with a multiplication. The multiplication is consistent with the bracket operation, i.e. $[e, f] = ef - fe$. The universal enveloping algebra carries the same information as the Lie algebra, but the additional multiplication structure opens for more flexibility in handling the object.

A quantum group is a sort of "deformation" of the universal enveloping algebra, denoted $U_q(\mathfrak{g})$. The index $q$ paramterizes the deformations and $q = 1$ corresponds to the original universal enveloping algebra. The quantum group $U_q(\mathfrak{sl}_2)$ is generated as an algebra by a symbol $t$ and its invers $t^{-1}$, in addition to the generators $e$ and $f$ of

$U(\mathfrak{g})$. The generators satisfies the relations:

$$tet^{-1} = q^2 e$$
$$tft^{-1} = q^{-2} f$$
$$[e, f] = \frac{t - t^{-1}}{q - q^{-1}}$$

Substituting $t = q^h$ and letting $q \to 1$, the relations are transformed into the ordinary relations for the Lie algebra $\mathfrak{sl}_2$:

$$[h, e] = 2e, \qquad [h, f] = -2f \qquad [e, f] = h$$

This can be justified by a straightforward computation using the linear approximation of the exponential function.

## Crystal basis theorem

Let $K = \mathbb{Q}(q)$ be the set of all rational functions $\frac{f(q)}{g(q)}$, where $f$ and $g$ are polynomials in a variable $q$. Let $A$ be the ring of rational functions in $q$ where the denominator $g(0) \neq 0$. Let $V$ be a $K$-vector space. A free $A$-module $L$ such that $K \otimes_A L \simeq V$ is called a lattice of $V$.

Let $M$ be an integrable $U_q(\mathfrak{sl})$-module, i.e. a module $M$ which admits a decomposition as a sum of eigenspaces of the operator $t \in U_q(\mathfrak{sl})$.

A crystal basis of $M$ is a pair $(L, B)$ where $L$ is a lattice of $M$ and $B$ is a basis of the $\mathbb{Q}$-vector space $L/qL$ such that $L$ and $B$ decompose similar to $M$ and such that $L$ and $B$ are respected by the Kashiwara operators $\tilde{e}$ and $\tilde{f}$. The last property defining a crystal basiss is that for $u, v \in B$, then $u = \tilde{e}v$ if and only if $v = \tilde{f}u$. The Kashiwara operators are operators on $M$ deduced from the generators $e$ and $f$.

Kashiwara´s theorem says that any integrable $U_q$-module has a crystal basis.

We shall illustrate the theory by considering a couple of examples.

A 2-dimensional representation $V = K^2$ of $U_q(\mathfrak{sl})$ with basis $\{v_1, v_{-1}\}$ over $K = \mathbb{Q}(q)$ is given by

$$e \cdot v_1 = 0 \quad f \cdot v_1 = v_{-1} \quad t^{\pm 1} \cdot v_1 = q^{\pm 1} v_1$$
$$e \cdot v_{-1} = v_1 \quad f \cdot v_{-1} = 0 \quad t^{\pm 1} \cdot v_{-1} = q^{\mp 1} v_{-1}$$

Relative to the basis $\{v_1, v_{-1}\}$ the action is given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

which is easily seen to respect the relations of the generators of $U_q(\mathfrak{sl}_2)$, given above. The basis $\{v_1, v_{-1}\}$ satisfies all requirements for a crystal basis, and the combinatorics of the basis is illustrated by the graph

$$v_1 \rightleftarrows v_{-1}$$

where the arrows to the right represents $\tilde{f}$ and arrows to the left represents $\tilde{e}$.

To explore more details in Kashiwara´s work on crystal bases we need to consider a more elaborate example. So we consider the $U_q(\mathfrak{sl}_2)$-module $W = V \otimes V$ where $V$ is as above.

A basis for $W$ as a vector space over $K$ is given by

$$\{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$$

The action of $U_q(\mathfrak{sl}_2)$ on $W$ is given by the coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, defined on the generators as follows:

$$\Delta(e) = e \otimes t^{-1} + 1 \otimes e$$
$$\Delta(f) = f \otimes 1 + t \otimes f$$
$$\Delta(t^{\pm 1}) = t^{\pm 1} \otimes t^{\pm 1}$$

The table gives the explicit details of the action of $e$, $f$ and $t$ on the basis elements:

| | $e$ | $f$ | $t$ |
|---|---|---|---|
| $v_1 \otimes v_1$ | 0 | $v_{-1} \otimes v_1$ $+qv_1 \otimes v_{-1}$ | $q^2 v_1 \otimes v_1$ |
| $v_1 \otimes v_{-1}$ | $v_1 \otimes v_1$ | $v_{-1} \otimes v_{-1}$ | $v_1 \otimes v_{-1}$ |
| $v_{-1} \otimes v_1$ | $q^{-1} v_1 \otimes v_1$ | $q^{-1} v_{-1} \otimes v_{-1}$ | $v_{-1} \otimes v_1$ |
| $v_{-1} \otimes v_{-1}$ | $qv_1 \otimes v_{-1}$ $+v_{-1} \otimes v_1$ | 0 | $q^{-2} v_{-1} \otimes v_{-1}$ |

The basis has some good properties, but some important details are missing, e.g. there is a serious problem for $q \to 0$.

Thus we have to modify the basis. Let

$$u_0 = qv_{-1} \otimes v_1 - v_1 \otimes v_{-1}$$
$$u_1 = v_1 \otimes v_1$$

An easy computation shows that $e \cdot u_0 = e \cdot u_1 = 0$, and $u_0$ and $u_1$ are eigenvectors for $t$ with weight (eigenvalues) 1 and $q^2$ respectively. Furthermore we have

$$f \cdot u_0 = 0, \quad f \cdot u_1 = v_{-1} \otimes v_1 + qv_1 \otimes v_{-1}$$

and $f \cdot u_1$ is a eigenvector for $t$ of weight 1. Iterating the action of $f$ we get an element

$$f^{(2)} \cdot u_1 = \frac{1}{q + q^{-1}} f^2 \cdot u_1 = v_{-1} \otimes v_{-1}$$

of weight $q^{-2}$. Thus we can weight-decompose $W$ as a sum of eigenspaces for the operator $t$ as

$$W = W(q^{-2}) \oplus W(0) \oplus W(q^2)$$

where $W(q^{-2})$ has rank 1, generated by $f^{(2)} \cdot u_1$, $W(0)$ has rank 2, with generators $u_0$ and $f \cdot u_1$ and finally the highest weight module $W(q^2)$ of rank 1 generated by $u_1$. A computation shows that

$$
\begin{aligned}
v_1 \otimes v_1 &= u_1 \\
v_{-1} \otimes v_1 &= \frac{q}{q^2+1} u_0 + \frac{1}{q^2+1} f \cdot u_1 \\
v_1 \otimes v_{-1} &= -\frac{1}{q^2+1} u_0 + \frac{q}{q^2+1} f \cdot u_1 \\
v_{-1} \otimes v_{-1} &= f^{(2)} \cdot u_1
\end{aligned}
$$

and

$$\mathcal{B} = \{u_0, u_1, u_2 = f \cdot u_1, u_3 = f^{(2)} \cdot u_1\}$$

is another basis for $W$. And this basis is perfect. It is made up by eigenvectors for $t$, we can let $q \to 0$ to get a basis for a 4-dimensional vector space over $\mathbb{Q}$, and it has the right properties with respect to the action of the Kashiwara operators $\tilde{e}$ and $\tilde{f}$. The Kashiwara operators act on the basis $\mathcal{B}$ by the rule

$$\tilde{e}(u_0) = \tilde{e}(u_1) = 0, \quad \tilde{e}(u_2) = u_1, \quad \tilde{e}(u_3) = u_2$$

$$\tilde{f}(u_0) = \tilde{e}(u_3) = 0, \quad \tilde{e}(u_1) = u_2, \quad \tilde{e}(u_2) = u_3$$

The basis $\mathcal{B}$ satisfies all requirements for a crystal basis, and the combinatorics of the basis is illustrated by the graph

$$u_1 \longleftrightarrow u_2 \longleftrightarrow u_3$$

together with a singleton $u_0$. The arrows to the right represents $\tilde{f}$ and arrows to the left represents $\tilde{e}$. The two components of the graph illustrate that $W$ is made up by two irreducible representations, one 3-dimensional generated by $u_1$, $u_2$ and $u_3$, and one 1-dimensional generated by $u_0$.