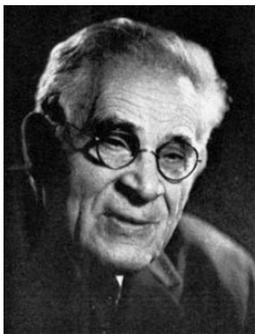




THE
ABEL
PRIZE

Gerd Faltings Abel Prize Laureate 2026

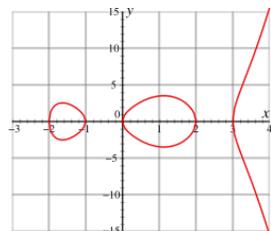
Faltings' theorem



Louis Mordell,
1888-1972

(Photo: The University of Manchester)

In 1922, Louis Joel Mordell proved that the rational solutions of a cubic equation defined over \mathbb{Q} form a finitely generated abelian group. In the same paper, he remarked that for curves of higher genus there might be only finitely many rational points. For several decades, this claim was known as the Mordell conjecture. It was not until the mid-80s the conjecture was finally proved to be correct. In his paper "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern", published in *Inventiones Mathematicae* in 1983, Gerd Faltings finally proved the conjecture and thereby turned the Mordell conjecture into Faltings' Theorem.



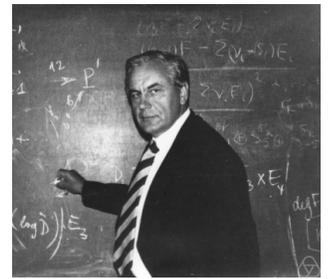
An example of a genus 2-curve. The curve is the zero set of the equation

$$y^2 = x(x+1)(x-2)(x+2)(x-3)$$

The general form of Faltings' Theorem is as follows: Let C be a smooth, projective curve of genus $g \geq 2$ defined

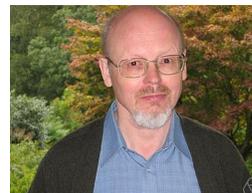
over a number field K . Then the K -points $C(K)$ of C is a finite set.

A major part of Faltings' proof of the Mordell conjecture is the proof of the Shafarevich conjecture. The Shafarevich conjecture was presented by the Soviet mathematician Igor Shafarevich at the International Congress of Mathematicians in Stockholm in 1962. He claimed that there should only be finitely many isomorphism classes of abelian varieties of a given dimension, with fixed polarization degree and defined over a field with good reduction outside a finite set of places.



Igor Shafarevich
1923-2017

(Photo: Konrad Jacobs, Erlangen)



Aleksei Parshin
1942-2022

(Photo: Wikipedia)

Assume for now that the Shafarevich conjecture is true. Let C be a curve over K and let P be a K -point on C . Aleksei Parshin showed in his doctoral thesis from 1968 how one can produce a finite cover $C_P \rightarrow C$ of bounded genus and ramified only over P . By a classical result of the Italian mathematician Michele de Franchis from 1913, there are only finitely many maps $C_P \rightarrow C$ for fixed C_P and C when C has genus greater



than or equal to 2.

By the Shafarevich conjecture, there are only finitely many curves C_p . Consequently, the number of K -points on C must also be finite. This bridge between the Shafarevich and Mordell conjectures is known as Parshin's trick. With Parshin's trick at hand, a proof of the Mordell conjecture is reduced to a proof of the Shafarevich conjecture.

An abelian variety is an algebraic variety which is also an abelian group, and the two structures are closely related. Abelian curves, i.e., abelian varieties of dimension 1, are known as elliptic curves

An isogeny between two abelian varieties E_1 and E_2 of the same dimension over a field k is a dense map $f : E_1 \rightarrow E_2$ mapping the identity of E_1 to the identity of E_2 . Alternatively, the map can be defined as a surjective group homomorphism with finite kernel.

What is left now is a proof of the Shafarevich conjecture, i.e., to show that there are only finitely many isomorphism classes of abelian varieties of a given dimension, with fixed polarization degree and defined over a field with good reduction outside a finite set of places. The second half of the statement, concerning polarization and reduction, is an important part of the conjecture, but for the big picture, focus on: finitely many classes of abelian varieties.

A key innovation of Faltings is a new notion of a height of an abelian variety. The construction is rather technical, but the consequence is important. Faltings shows that if one requires bounded height, the set of abelian varieties is finite. He also shows that internally in an isogeny class the height is bounded.

Combining the two results it follows that for an abelian variety A there are only finitely many isomorphism classes of abelian varieties isogenous to A .

For an abelian group A and a prime p the p -adic Tate module of A , named after Abel Prize Laureate 2010 John Torrence Tate Jr., is defined as the inverse limit

$$T_p(A) = \varprojlim A[p^n]$$

where $A[p^n]$ is the p^n torsion of A , i.e. the kernel of the map given by multiplication by p^n , and the inverse limit is computed over the inverse system given by multiplication by p ; $A[p^{n+1}] \rightarrow A[p^n]$. Thus, the Tate module of A encodes the p -torsion of A .

Let K be a field of characteristic $\text{char}(K) \neq p$, and A and B two abelian varieties over K . The Tate Conjecture, proved for finite fields by Tate in 1966 and for number fields by Faltings in 1983, establishes an isomorphism

$$\text{Hom}_K(A, B) \otimes \mathbb{Z}_p \simeq \text{Hom}_{G_K}(T_p(A), T_p(B))$$

where $G = \text{Gal}(\bar{K}/K)$.

The remaining step in the big picture of the proof of the Shafarevich conjecture is to show that there are only finitely many isogeny classes of abelian varieties. The crucial point is the Tate conjecture, which Faltings proved to be true for number fields. The Tate conjecture gives a close correspondence between isogenies of A and G -representations of the ℓ -adic Tate module of A . Finiteness of the set of isogenies corresponds to semisimplicity of the action of G on the Tate module. By combining some classical results about representations, Faltings proved that the number of isogeny classes, and also the number of isomorphism classes, of abelian varieties with good reduction outside a finite set of places of K is finite, thereby proving the Shafarevich conjecture.



Gerd Faltings in 2005
(Photo: Renate Schmind, MFO)

