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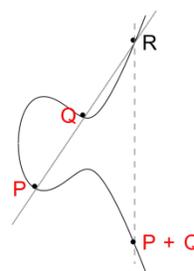
Gerd Faltings
Abel Prize Laureate 2026

From Mordell's Conjecture to Faltings' Theorem

Historically, the field of mathematics is founded on two pillars: number theory and geometry. The core of number theory is the natural numbers, and their extension to rational numbers. In primary school, children learn about addition and multiplication, and the connection between them; multiplication can be considered as repeated addition. But this naive approach hides a more mysterious side of the numbers. An illustrating example of the complexity of mixing addition and multiplication is the ancient question of whether it is possible to add two perfect squares and obtain a perfect square as their sum. The question is closely related to the Pythagorean theorem, relating the squares of the sides of a right triangle. The number theoretical question behind the theorem concerns integer solutions (a, b, c) to the equation $a^2 + b^2 = c^2$, often denoted Pythagorean triples,

It has been known since ancient times that there are infinitely many Pythagorean triples, the easiest example is reflected in the carpenter's 3-4-5-method, based on the fact that $3^2 + 4^2 = 5^2$ and used by carpenters to decide if an angle is right. A probabilistic explanation for the existence of infinitely many Pythagorean triples is that the number of perfect squares is a sufficiently dense subset of the natural numbers. Thus, the set of natural numbers which are a sum of two perfect squares and the set of perfect squares have infinite overlap.

If we proceed to perfect cubes, the frequency of such numbers within the set of natural numbers is far more sparse than the squares, and it is not obvious that the analogous problem to that of finding Pythagorean triples now has a positive solution. In fact, by Fermat's last theorem, proved by Abel Laureate Andrew Wiles, it is not possible at all. Fermat's last theorem concerns integer solutions of the equation $x^d + y^d = z^d$ for $d \geq 3$. Mordell's conjecture from 1922 suggests that for all high degree equations of a certain form, the equations has only finitely many rational solutions.



An elliptic curve
(Illustration by Dr Jennifer
Balakrishnan, Oxford)

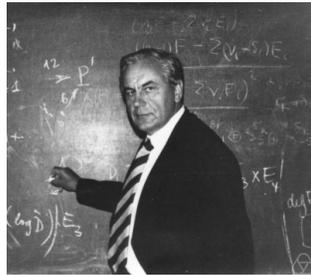
The solution of an equations of degree 3 has a geometrical interpretation as the points on an elliptic curve. An elliptic curve has genus 1, referring to a certain characteristic of the shape of the curve. Curves of genus ≥ 2 are geometrically more complex and defined by equations of even higher degree. Among the points on a curve there might be some for which the coordinates are rational numbers, referred to as rational points. The Mordell conjecture says that there are only a finite number of rational points on a curve of genus ≥ 2 . Mordell himself was only able to prove a slightly less strict result; that there might be infinitely many rational points on an elliptic curve, but only a finite number of them are needed to construct the rest. The construction refers to the fact that elliptic curves are equipped with a natural addition operation, making the set of points on the curve



into what mathematicians call an abelian group. Abelian groups are named after Niels Henrik Abel, from whom the Abel Prize takes its name.

The Mordell conjecture for curves of genus ≥ 2 remained unproved for more than 60 years. During this period of time, some results with connections to the conjecture were established, among them a conjecture of Shafarevich concerning finiteness of a family of curves. By a result referred to as Parshin's trick there is a close connection between the above mentioned family of curves and a curve which lies behind that family. In the early 80s it was known that a proof of the Shafarevich conjecture would automatically give a proof of the Mordell conjecture.

In 1983, Gerd Faltings succeeded in proving the Shafarevich conjecture. A main step in his proof is the introduction of what is now called the Faltings height of a curve. Faltings shows a finiteness result for the number of certain curves of bounded Faltings height and also a boundedness result for the Faltings height itself. By combining the two, a finiteness result for the number of the curves in question is obtained, thereby proving the Shafarevich conjecture. By using Parshin's trick it follows that the Mordell conjecture is no longer a conjecture, it is now a result called Faltings' Theorem.



Igor Shafarevich
1923-2017
(foto: Konrad Jacobs, Erlangen)

